The Derivation and Study of the Nonlinear Schrödinger Equation for Long Waves in Shallow Water Using the Reductive Perturbation and Complex Ansatz Methods

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Abstract: In this paper, the water wave flow problem for an incompressible and inviscid fluid of constant depth is studied under the influence of acceleration of gravity and surface tension. The nonlinear Schrödinger (NLS) equation and the dispersion relation are derived from the nonlinear shallow water equations by using the reductive perturbation technique, which differs from the derivations of the same problem illustrated in previous works. The complex ansatz method is presented for constructing exact traveling wave solutions of NLS equation and from them the physical variables of the water wave problem are obtained. Depending on the Ursell parameter, the diagrams are drawn to illustrate the behavior of the solutions of NLS equation, free surface elevation and velocity of the model. The results indicate that the solutions of this problem are in the form of envelope traveling solitary waves where the Ursell parameter affects the wave profile significantly. It is concluded the chosen methods of solutions are sufficiently accurate to demonstrate that the conservation of power is satisfied except at very few spots where it fluctuates about zero by the orders of $10^{-14} - 10^{-18}$ depending on the values of the Ursell parameter.

Keywords: shallow water equations; reductive perturbation method; nonlinear Schrödinger equation; complex ansatz method

1 Introduction

Many phenomena in physics and other applied fields are described by nonlinear partial differential equations (PDEs). To understand the physical situation of these phenomena in nature, we need to find the exact solutions of the PDEs, which have become one of the most important topics in mathematical physics. In the study of equations modeling wave phenomena, one of the fundamental objects of study is the traveling wave solution, meaning a solution of constant form moving with a fixed velocity. Of particular interest are three types of traveling waves: solitary waves, periodic waves and kink waves [1-9].

The problem of nonlinear waves propagating in plasma and fluid can be described by the KdV, nonlinear Schrödinger, Davy-Stewartson equations and others. These last equations are derived by multiple scale, reductive perturbation methods and other asymptotic methods [10-19].

Due to its central importance to the theory of quantum mechanics, the nonlinear equation of Schrödinger type has a great interest. They arise in many physical problems, including nonlinear water waves, ocean waves, waves in plasma, propagation of heat pulses in a solid self trapping phenomenon in nonlinear optics, nonlinear waves in a fluid-filled viscoelastic tube, and various nonlinear instability phenomena in fluids and plasma, and are of importance in the development of solitons and inverse scattering transform theory [1-4].

In the past the exact solution of Schrödinger type was obtained by converting these equations into real forms through some transformations and then using some methods such as Jacobi elliptic expansion, tanh-function method, Cole-Hopf transformation, Hirota bilinear method, inverse scattering method and so on. Recently, direct methods are proposed to obtain the exact wave solutions of these equations such as complex tanh-function method, complex hyperbolic-function method, sub-ODE method, complex ansatz method, complex Jacobi elliptic method and others [1, 2, 20-30].

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In this paper we study the irrotational incompressible flow of a shallow layer of inviscid fluid moving under the influence of gravity as well as surface tension. Previously Dullin et al studied this case without surface tension, which in the shallow water approximation leads to the Camassa-Holm (CH) equation [7]. Also Dullin et al derived the CH as a shallow water wave equation in an asymptotic expansion that extends one order beyond the Korteweg-de Vries equation (KdV) and they showed that CH is asymptotically equivalent to the fifth-order integrable equation in the KdV hierarchy by using the Kodama transformation [8, 9]. Lizuka and Wadati applied the reductive perturbation method to the two-dimensional Rayleigh-Tayler problem for the interface between light and heavy inviscid incompressible fluids by including the effect of surface tension and derived stable and unstable NLS equations and nonlinear diffusion equation [10]. Abourabia et al. studied the problem of water waves that propagating at the interface between two inviscid fluids by using the multiple scale method to obtain the NLS equation [19].

The paper consists of the following. In section 2, we describe and formulate the water wave problem, and then study its governing equations. In section 3, the NLS equation and dispersion properties for the nonlinear shallow water equations are derived by the reductive perturbation technique. In section 4, the complex ansatz method is applied to the obtained NLS equation to find the solutions of the problem. Section 5 is devoted for the conclusions.

2 Theoretical formulation of the basic equations for the problem

Let us consider a shallow water wave propagating in a finite depth water that has a free surface. So we assume an incompressible, inviscid fluid of constant undisturbed depth \( h_0 \) and constant density \( \rho \) with acceleration gravity \( g \) and surface tension. We assume also that \( x - y \) plane is the undisturbed free surface with the \( z \)-axis positive upward. The free surface elevation above the undisturbed depth is \( \eta(x, y, t) \), so that the free surface is at \( z = h_0 + \eta \) and the horizontal flat bottom is at \( z = 0 \). Denote by \( u \) and \( v \) the horizontal and vertical velocity components, respectively.

The velocity potential \( \varphi(x, y, z, t) \) and the free surface elevation \( \eta(x, y, t) \) are governed by the following Laplace equation and boundary conditions:

\[
\nabla^2 \varphi = \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, \quad 0 < z < h_0 + \eta, \quad -\infty < x, y < \infty. \tag{1}
\]

The dynamic and kinematic conditions at the free surface are

\[
\varphi_t + g \eta + \frac{1}{2} (\nabla \varphi)^2 - \frac{\Gamma}{\rho} (\eta_{xx} + \eta_{yy}) = 0, \quad \text{on} \quad z = h_0 + \eta, \tag{2}
\]

\[
\eta_t + \eta_x \varphi_x + \eta_y \varphi_y - \varphi_z = 0, \quad \text{on} \quad z = h_0 + \eta. \tag{3}
\]

The boundary condition at the flat bottom is

\[
\varphi_z = 0, \quad \text{on} \quad z = 0, \tag{4}
\]

here and hereafter, the subscripts \( x, y, z \) and \( t \) denote the partial derivatives.

These equations for a fluid will be written in a nondimensional form by introducing the following flow variables which is based on different length scales, a typical horizontal wave length \( \lambda_0 \) and a typical vertical length \( h_0 \) [1, 2, 8, 9, 31]

\[
(x, y) = \lambda_0 (\bar{x}, \bar{y}), \quad z = h_0 \bar{z}, \quad t = \frac{\lambda_0}{c_0} \bar{t}, \quad \eta = a \eta, \quad \varphi = \frac{a \lambda_0 c_0}{h_0} \bar{\varphi}, \tag{5}
\]

where \( a \) is the surface wave amplitude, \( c_0 = \sqrt{g h_0} \) is a typical horizontal velocity (shallow water wave speed), and the bars refer to the nondimensional variables.

We also introduce two fundamental parameters to characterize the nonlinear shallow water waves namely \( \varepsilon = a/h_0 \) as a measure of nonlinearity and \( \delta = h_0^2/\lambda_0^2 \) as a measure of dispersion (shallowness) [1].

Introducing (5) into (1)-(4), the governing equations can be expressed in nondimensional form, dropping the bars, as follows

\[
\delta (\varphi_{xx} + \varphi_{yy}) + \varphi_{zz} = 0, \quad 0 < z < 1 + \varepsilon \eta, \tag{6}
\]

\[
\varphi_t + \eta + \frac{\varepsilon}{2} (\varphi_x^2 + \varphi_y^2) + \frac{\varepsilon}{2 \delta} \varphi_z^2 - \sigma \delta (\eta_{xx} + \eta_{yy}) = 0, \quad \text{on} \quad z = 1 + \varepsilon \eta, \tag{7}
\]

\[
\delta (\eta_t + \varepsilon (\eta_x \varphi_x + \eta_y \varphi_y)) - \varphi_z = 0, \quad \text{on} \quad z = 1 + \varepsilon \eta, \tag{8}
\]

\[
\varphi_z = 0, \quad \text{on} \quad z = 0. \tag{9}
\]
where \( \sigma = \Gamma / h_0 \rho c_0^2 \) is the Bond number, which is the ratio of surface tension force \( \Gamma \) to the body force, [8, 9, 19]. A low Bond number indicates that the system is relatively unaffected by surface tension effects, while a high Bond number indicates that surface tension dominates. Intermediate Bond numbers indicate a non-trivial balance between the two effects.

The basic equations (6)-(9) are too complicated to handle in order to obtain a compact evolution equation for one physical variable without approximations. So, we can begin the assumption that \( \delta \) is small, which might be interpreted as the characteristic feature of the shallow water theory. It follows that \( \phi \) could be expanded in terms of \( \delta \) without any assumption yet about \( \epsilon \) and write

\[
\phi = \phi_0 + \delta \phi_1 + \delta^2 \phi_2 + \cdots
\]  

(10)

Substituting (10) into (6), we find that at

\[
O(\delta^0) : \phi_{0,z,z} = 0,
\]

(11)

\[
O(\delta^1) : \phi_{1,z,z} + \phi_{0,y,y} + \phi_{0,x,x} = 0,
\]

(12)

\[
O(\delta^2) : \phi_{2,z,z} + \phi_{1,y,y} + \phi_{1,x,x} = 0.
\]

(13)

Integrating (11) twice w. r. t. \( z \) and by using (9), we get \( \phi_0 = c_1(x, y, t) \), which indicates that the horizontal velocity components are independent of the vertical coordinate [1].

In this problem of water waves, the motion may be taken to be irrotational, which physically means that the individual fluid particles don’t rotate. Mathematically, this implies that the vorticity vanishes, \( \nabla \cdot \vec{u} = 0 \). So that there exists a single valued velocity potential, \( \vec{u} = \nabla \phi \), or equivalently

\[
\phi_{0,x} = u, \quad \phi_{0,y} = v.
\]

(14)

Hence, integrating (12) w. r. t. \( z \) to obtain

\[
\phi_{1,z} = -z(u_x + v_y) + c_2(x, y, t).
\]

(15)

Using Eq. (9) the integration constant \( c_2(x, y, t) \) equals zero, and integrating the resulting equation we obtain

\[
\phi_1 = -\frac{z^2}{2} (u_x + v_y),
\]

(16)

where the integration constant is assumed equal to zero.

Substituting (16) into (13) and integrating the resulting equation twice w. r. t. \( z \) we get

\[
\phi_2 = -\frac{z^4}{24} \left( (\nabla^2 u)_x + (\nabla^2 v)_y \right)
\]

(17)

Using Eqs. (15)-(17), we substitute (10) into the free surface boundary conditions (7)-(8) considering that these conditions retain all terms up to order \( \delta, \epsilon \) in Eq. (7) and \( \delta, \delta^2, \delta \epsilon \) in (8). It turns out that

\[
\phi_{0,t} + \eta + \frac{\epsilon}{2} (u^2 + v^2) - \frac{\delta}{2} (u_{t,x} + u_{t,y}) - \sigma \delta (\eta_{x,x} + \eta_{y,y}) = 0,
\]

(18)

\[
\frac{\delta}{2} (\eta_{t} + (1 + \epsilon \eta) (u_x + v_y) + \epsilon (u \eta_y + v \eta_x)) - \frac{\delta^2}{6} \left( (\nabla^2 u)_x + (\nabla^2 v)_y \right) = 0.
\]

(19)

Using (14), differentiation of (18) first w. r. t. \( x \) and then w. r. t. \( y \) gives the following two equations

\[
u_t + \eta_x + \epsilon (u u_x + v v_x) - \frac{\delta}{2} (u_{x,t} + u_{y,t}) - \sigma \delta (\eta_{x,x,x} + \eta_{x,y,y}) = 0,
\]

(20)

\[
v_t + \eta_y + \epsilon (u u_y + v v_y) - \frac{\delta}{2} (u_{y,t} + u_{x,t}) - \sigma \delta (\eta_{x,x,y} + \eta_{x,y,y}) = 0.
\]

(21)

Using the fact that \( \phi_0 \) is irrotational, that is \( u_y = v_x \), we obtain the nondimensional shallow water equations

\[
u_t + \eta_x + \epsilon (u u_x + v u_x) - \frac{\delta}{2} (\nabla^2 u)_t - \sigma \delta (\nabla^2 \eta)_x = 0,
\]

(22)

\[
v_t + \eta_y + \epsilon (u u_y + v v_y) - \frac{\delta}{2} (\nabla^2 v)_t - \sigma \delta (\nabla^2 \eta)_y = 0,
\]

(23)
\[ \eta_t + (u (1 + \varepsilon \eta))_x + (v (1 + \varepsilon \eta))_y - \frac{\delta}{6} \left( \left( \nabla^2 u \right)_x + \left( \nabla^2 v \right)_y \right) = 0. \] (24)

In the following study we consider the one-dimensional case of the above equations and the nonlinear parameter \( \varepsilon \) tends to zero for \( \delta \) fixed, so that Eqs. (22)-(24) reduce to

\[ \delta^{-1} \left( u_t + \eta_x \right) + U_r u u_x - \frac{1}{2} u_{x,x} + \eta \eta_x = 0, \] (25)
\[ \delta^{-1} \left( \eta_t + u_x \right) + U_r (u \eta)_x - \frac{1}{6} u_{x,x} = 0. \] (26)

We take these two equations as the starting point to apply the reductive perturbation method, and taking that

\[ U_r = \frac{\varepsilon}{\delta} = \frac{a \lambda^3}{h_0^3}, \] (27)

defined as the Ursell parameter [32], which and is derived from the Stokes’ perturbation series for nonlinear periodic waves in the long-wave limit of shallow water. This parameter characterizes the relative role of nonlinearity and dispersion: small values of \( U_r \) correspond to the almost linear dispersive waves; while large values of \( U_r \) correspond to the nonlinear nondispersive waves [33].

### 3 Derivation of NLS Equation Using Reductive Perturbation Technique

Here, we carry out the standard reductive perturbation analysis [10, 14, 15] for the nonlinear shallow water equations (25) and (26) to obtain the NLS equation which governs the behavior of the one dimensional case of shallow water waves through the constant water depth.

In this technique, the independent variables are scaled according to the Gardner-Morikawa transformation [10]

\[ \xi = \varepsilon (x - v_g t), \quad \tau = \varepsilon^2 t, \] (28)

where \( \varepsilon \) is a small \((0 < \varepsilon < 1)\) expansion parameter defined before and \( v_g \) is the group velocity of the water wave that will be determined later.

The original equations (25) and (26), are transformed according to

\[ \frac{\partial}{\partial t} \Rightarrow \frac{\partial}{\partial t} - \varepsilon v_g \frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x} \Rightarrow \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \xi}, \] (29)

to yield

\[ \delta^{-1} \left( u_t - \varepsilon v_g u_x + \varepsilon^2 u_{xx} + \eta_x + \eta \eta_x \right) + U_r u (u_x + \varepsilon u_x) - \frac{1}{2} \left( u_{xx} + \varepsilon \left( 2 u_t, x, \xi - v_g u_x, x, \xi \right) + \varepsilon^2 \left( 2 u_t, x, \xi - v_g u_x, x, \xi \right) + \varepsilon^3 \left( 2 u_t, x, \xi - v_g u_x, x, \xi \right) + \varepsilon^4 u_{xx}, x, \xi \right) - \sigma \left( \eta_x, x, x \right) \]
\[ + 3 \varepsilon \eta_x, x, x + 3 \varepsilon^2 \eta_x, x, x + \varepsilon^3 \eta_x, x, x + \varepsilon^4 \eta_x, x, x = 0, \] (30)
\[ \delta^{-1} \left( \eta_t - \varepsilon v_g \eta_x + \varepsilon^2 \eta_{xx} + \eta_x + \varepsilon \eta_x \right) + U_r (u \eta_x + \varepsilon u_x + \varepsilon \eta_x + \varepsilon \eta_x) \]
\[ - \frac{1}{2} \left( u_{xx}, x, x + 3 \varepsilon u_x, x, x + 3 \varepsilon^2 u_x, x, \xi + \varepsilon^3 u_x, x, \xi + \varepsilon^4 u_x, x, \xi \right) = 0, \] (31)

thus the functions \( u(x, t) \) and \( \eta(x, t) \) are now treated as functions of the variables \((\xi, \tau)\).

Using the small perturbation parameter \( \varepsilon \), the dependent physical variables can be expressed as

\[ \eta = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \varepsilon^n E^m \eta^{mn} (\xi, \tau), \] (32)
\[ u = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \varepsilon^n E^m u^{mn} (\xi, \tau), \] (33)

where \( E = i (k x - \omega t) \), \( k \) denotes the wave number and \( \omega \) the angular frequency. Since the physical quantities \( \eta \) and \( u \) are real, the relations

\[ \eta^{(-m)n} = \eta^{* mn}, \quad u^{(-m)n} = u^{* mn}, \] (34)

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are satisfied, where the asterisk denotes the complex conjugate.

Applying the above series expansion to Eqs. (30)-(31) and equating each coefficient of different powers of \( \varepsilon \) to zero, we get, for the first order of \( \varepsilon \) and \( n = 1 \), the first order quantity \( \eta^{11} \) in terms of \( u^{11} \) as

\[
\eta^{11} = \frac{k (6 + k^2 \delta)}{6 \omega} u^{11},
\]

(35)

and the dispersion relation

\[
\omega^2 = \frac{k^2 (6 + k^2 \delta) (1 + k^2 \delta \sigma)}{6 + 3 k^2 \delta},
\]

(36)

for the shallow water equations, from it we can obtain the phase velocity

\[
v_p = \frac{\omega}{k} = \sqrt{\frac{(6 + k^2 \delta) (1 + k^2 \delta \sigma)}{6 + 3 k^2 \delta}}.
\]

(37)

For \( n = 1 \) and \( m = 2, 3 \), we get

\[
\eta^{21} = \eta^{31} = u^{21} = u^{31} = 0.
\]

(38)

For the order of \( \varepsilon^2 \) with \( m = 0, 1, 2 \), we obtain the following equations

\[
\eta^{01} = u^{01} = 0,
\]

(39)

\[
\eta^{12} = \frac{k (6 + k^2 \delta)}{6 \omega} \frac{i (k v_g (6 + k^2 \delta) - 3 (2 + k^2 \delta) \omega)}{6 \omega^2} u^{11}_x,
\]

(40)

\[
v_g = \frac{k (12 + k^2 \delta (4 + 24 \sigma + k^2 \delta (1 + 2 (6 + k^2 \delta) \sigma)) \omega)}{3 (2 + k^2 \delta)^2 \omega} \frac{\partial \omega}{\partial k},
\]

(41)

\[
\eta^{22} = \frac{- (2 + k^2 \delta) (9 + k^2 \delta (15 + 2 k^2 \delta)) U_r}{6 k^2 (-2 + (6 + k^2 \delta (5 + 2 k^2 \delta) \sigma) \omega)} (u^{11})^2,
\]

(42)

\[
u^{22} = \frac{- (6 + k^2 \delta) (3 + k^2 \delta (1 + (9 + 4 k^2 \delta) \sigma)) U_r}{6 k^2 (-2 + (6 + k^2 \delta (5 + 2 k^2 \delta) \sigma) \omega)} (u^{11})^2.
\]

(43)

where the compatibility condition (41) is interpreted as the group velocity of the shallow water wave equations.

The graphs of this paper will be drawn to describe the physical situation of the model when we use suitable values for the parameters as \( h = 1 \) m, \( \rho = 1000 \text{ kg/m}^3 \), \( \Gamma = 0.07197 \text{ N/m} \), \( \sigma = 7.34 \times 10^{-6} \), \( g = 9.8 \text{ m/s}^2 \). The dependence of \( \omega, v_p \) and \( v_g \) on the wave number \( k \) are shown in Fig. 1.

![Graphs of dispersion properties](image)

**Figure 1:** The graphs of the dispersion properties against \( k \) at \( \varepsilon = \delta = 0.03 \).

In Fig. 1 we found that the phase velocity represented by the dashed line is greater than the group velocity represented by the solid line \( (v_g > v_p) \) which is satisfied for gravity waves, while the opposite is true for capillary waves, see [3]. This means that the dispersion relation for the shallow water equations is normal. We also noticed that the phase and group velocities decrease as the wave number \( k \) increases which is consistent with long waves that travel faster than shorter ones.

For the order of \( \varepsilon^3 \) with the zeroth order harmonic mode of carrier wave \( (m = 0) \), we obtain

\[
\eta^{02} = \frac{\delta (k v_g (6 + k^2 \delta) + 3 \omega) U_r}{3 (-1 + v_g^2 \omega)} (u^{11})^2.
\]

(44)
in Fig. 2. The exact solutions of Eq. (46) will be discussed in the following section.

Now, by using the above derived equations for the order of \( \epsilon^3 \) with the first harmonic mode \( (m = 1) \), we get a closed evolution equation for \( u^{11} \):

\[
i u^{11} + p u^{11}_{\xi} + q \left| u^{11} \right|^2 u^{11} = 0,
\]

which is the well-known envelope nonlinear Schrödinger equation; it describes the evolution of the envelope of the modulated wave group. According to the stability criterion established in [1, 2, 14, 19], the wave solutions of this equation are stable if \( pq < 0 \) or unstable if \( pq > 0 \). To investigate these effects in more detail, we plot the product \( pq \) with \( k \) and \( \delta \) as in Fig. 2. The exact solutions of Eq. (46) will be discussed in the following section.

The dispersion coefficient \( p \) satisfies the relationship

\[
p = \frac{1}{2} \frac{\partial^2 \omega}{\partial k^2},
\]

and the nonlinearity coefficient \( q \) reads

\[
q = \left( \frac{1}{(v_c^2 - 1)} (\frac{2}{1 + \epsilon^2} (6 + k^2 \delta (5 + 2k^2 \delta)) \sigma \omega (6 + k^2 \delta) [1 + \epsilon^2 (2 + 2k^2 \delta)] \omega^2 \right) \times \left( \delta (4k^{12} \delta^2 \sigma + 2k^{12} \delta^2 \sigma (2 + 31 - 2v_c^2 \sigma)) + k^{10} \delta^3 \sigma (59 + 333 \sigma) \right) + 6k^{11} v_c \delta \sigma^2 \omega + 54k^5 v_c \delta \sigma^2 (5 + 12 \omega) + 18k^9 v_c \delta \sigma (2 + 17 \omega) \omega - 216(v_c^2 - 1) \omega^2 + 72k v_c \delta (3 \sigma - 1) \omega^3 + 36k^3 (3 - 3v_c^2) + 25(1 + 15 \sigma - 2v_c^2 (2 + 3 \sigma)) \omega^2 + 36k^3 v_c \delta \omega^2 (1 - 2 \omega (6 \sigma + (2 \omega))) + 36k^3 v_c \delta \omega (6 \sigma + (18 \sigma + 5 \omega)) + 12k^4 \delta (6 \sigma + (72 \sigma + \delta (8 + 9 \sigma (9 + 2 \sigma)) \omega^2 - 2v_c^2 (3 \sigma + 18 \sigma + (6 \sigma + (1 + 2 \sigma) \omega^2))) + 3k^6 \delta^2 (-11 + 228 \sigma + 252 \sigma^2 + 2 \delta (2 + 6 \sigma (30 \sigma^2 \omega^2 - v_c^2 (4 + 4 \omega^2 + 2 \sigma (54 + 5 \omega + 23 \delta \omega^2))) + k^8 \delta^3 (-v_c^2 (1 + 12 \sigma (6 + 21 \sigma + \delta \omega^2))) + 3(-1 + 4 \sigma (25 + 63 \sigma + 3 \delta (1 + 2 \sigma) \omega^2))) \right) U_c^2.
\]

In the following figures we study the stability of the NLS equation:

![Figure 2: The graphs for the variation of \( pq \) at \( \epsilon = 0.03 \).](image_url)

We can see from Fig. 2(a) that the wave solution of the NLS equation is stable at all values of the wave number \( k \) for \( \delta = 0.03 \), but in Fig. 2(b) the parameter space \( k-\delta \) is divided into two regions, the wave solution is unstable for very small values of \( k \) and \( \delta \) \((0 < k \leq 10^{-2}, 0 < \delta \leq 10^{-1} \)) while at larger values, the wave solution is stable. It means that there are two types of the soliton solutions of the NLS equation.

From the above results (35)-(43) we can obtain the physical solutions of the shallow water wave problem for first and second harmonics as

\[
\eta = \epsilon k (6 + k^2 \delta) (e^{-E} + e^{E}) U^{11}_{\text{low}} + \epsilon^2 U_c (u^{11})^2 \left( \frac{(2k^4 + 6k^4 \delta (5 + 2k^2 \delta)) (e^{-2E} + e^{2E})}{-6k^2 (2 + 6k^4 \delta (5 + 2k^2 \delta) \omega^2)} + \frac{\delta (6k v_c k^3 v_c \delta + 3 \omega)}{3(v_c^2 - 1) \omega} \right),
\]

\[
u = \epsilon (e^{-E} + e^{E}) U^{11}_{\text{low}} + \epsilon^2 U_c (u^{11})^2 \left( \frac{(6 + k^2 \delta) (3 + k^2 \delta (1 + (9 + 4k^2 \delta)) (e^{-2E} + e^{2E}))}{-6k^2 (2 + 6k^4 \delta (5 + 2k^2 \delta) \omega^2)} + \frac{\delta (6k v_c k^3 v_c \delta + 3 \omega)}{3(v_c^2 - 1) \omega} \right).
\]

Note that these expansions have been obtained symbolically by using the formal computation software "Mathematica".

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3.1 Conservation law for NLS equation

Zakharov and Shabat [1] proved that the NLS equation has infinite number of Polynomial conservation laws. These laws are somewhat similar to those already proved for the KdV equation [1, 2, 19].

Using the relation (35), we replace \( u^{11} \) in Eq. (46) by \( \eta^{11} \). Then we multiply the resulting equation by \( \eta^{11} * \) and its complex conjugate by \( \eta^{11} \), and hence subtracting the latter from the former to obtain the conservation of potential energy with respect to the elevation (amplitude) of the water wave which satisfy the continuity equation as follows:

\[
W \equiv i \rho_r + J_\xi = 0,
\]

where \( \rho = \eta^{11} \cdot \eta^{11} \) is the energy density and \( J = p \left( \eta^{11} \cdot \eta^{11} - \eta^{11} \cdot \eta^{11} \right) \) is the energy current density. We will discuss the continuity equation \( W \) in the following section.

4 The exact solutions of NLS equation using the complex Ansatz method

In this section, without transforming to real and imaginary parts, a complex ansatz method is presented to derive the exact complex traveling wave solutions of NLS equation [30].

For a given partial differential equations (PDEs)

\[
f(u, u_x, u_t, u_{x,x}, \cdots) = 0,
\]

Using the complex transformation \( \zeta = i (K \xi - \Omega \tau) \), Eq. (52) becomes an ordinary differential equation (ODEs)

\[
g(u, i K u_\zeta, -i \Omega u_\zeta, -K^2 u_\zeta, \zeta, \cdots) = 0,
\]

the proposed solutions could be taken in the form

\[
u(\zeta) = \sum_{i=0}^{l} a_i F^i(\zeta) + \sum_{i=1}^{l} b_i F^{i-1}(\zeta) G^i(\zeta),
\]

where \( a_i, b_i \) are constants to be determined and the integer \( l \) is determined by balancing the linear term of highest order with the nonlinear term.

The coupled Riccati equations,

\[
F_\zeta(\zeta) = -F(\zeta) \; G(\zeta) \; \text{and} \; G_\zeta(\zeta) = 1 - G^2(\zeta),
\]

admits two types of solutions as

\[
F(\zeta) = \pm \sec h(\zeta), \; G(\zeta) = \tanh(\zeta) \; \text{with the relation} \; G^2 = 1 - F^2,
\]

and

\[
F(\zeta) = \pm \csc h(\zeta), \; G(\zeta) = \coth(\zeta) \; \text{with the relation} \; G^2 = 1 + F^2,
\]

Then substituting (54) into (53), using (55) repeatedly with (56) or (57), and setting each coefficient of \( F^i \) and \( G F^i \) to zero yields a set of algebraic equations for \( a_i, b_i, K, \Omega \). Solving them gives the exact solutions for Eq. (55), accordingly the exact solutions of NLS equation (46) can be obtained. The ODE form of this equation is

\[
\Omega u_\zeta^{11} - pK^2 u_\zeta^{11} + qu^{112} u^{111} = 0,
\]

and the balancing between \( u_\zeta, \zeta^{11} \) and \( u^{112} \) \( (u^{11})^* \) yields \( l = 1 \). This suggests the following form of solution (56)

\[
u = a_0 + a_1 F + b_1 G.
\]

Using the relation in Eq. (56) with \( F^* = F, \; G^* = -G \), substituting (59) into (58) and equating all coefficients of \( F^i \) and \( G F^i \) to zero, we obtain

\[
q a_0^3 - q a_0 b_1^2 = 0, \; -\Omega a_1 + 2 q a_0 a_1 b_1 = 0, \; 3 q a_0 a_1^2 + \Omega b_1 + q a_0 b_1^2 = 0, \; -K^2 p a_1 + 3 q a_0^2 a_1 \\
- q a_1 b_1^2 = 0, \; 2K^2 p a_1 + q a_1^3 + q a_1 b_1^2 = 0, \; q a_0^2 b_1 - q b_1^3 = 0, \; 2K^2 p b_1 + q a_1^3 b_1 + q b_1^3 = 0.
\]
Solving these algebraic equations gives the following cases

Case 1, 2:

\[ \Omega = -2K^2p, \quad a_0 = \mp i K \sqrt{\frac{2p}{q}}, \quad a_1 = 0, \quad b_1 = -a_0, \quad (61) \]

Case 3, 4:

\[ \Omega = 2K^2p, \quad a_0 = \mp i K \sqrt{\frac{2p}{q}}, \quad a_1 = 0, \quad b_1 = a_0. \quad (62) \]

Therefore, we obtain the following family of complex exact traveling wave solutions for NLS equation as

\[ u_{11}^{12} = i K \sqrt{\frac{2p}{q}}(\mp 1 \pm \tanh (i K (\xi + 2Kp\tau))), \quad (63) \]

\[ u_{11}^{34} = \mp i K \sqrt{\frac{2p}{q}}(1 + \tanh (i K (\xi - 2Kp\tau))). \quad (64) \]

In terms of these solutions of \( u^{11} \), we can obtain the physical variables of the water wave problem in Eqs. (49) and (50).

To show the properties of the exact solutions for the NLS equation and the shallow water wave equations, we take some of these solutions as illustrative examples and draw their graphs as in Figs. 3, 4, 5 and 6.

(a) \( \varepsilon = 0.015, \delta = 0.03, U_r = 0.5 \)

(b) \( \varepsilon = 0.03, \delta = 0.035, U_r = 0.86 \)

Figure 3: The graphs for the solutions of NLS equation \( u_{11}^{11} \) at different values of Ursell parameter.

(c) \( \varepsilon = 0.015, \delta = 0.03, U_r = 0.5 \)

(d) \( \varepsilon = 0.03, \delta = 0.035, U_r = 0.86 \)
From the above figures we notice that the imaginary parts of solutions are in the form of bright and dark solitons, see Figs. 3(a), 4(e) and 5(i). Their amplitudes decrease as the Ursell parameter increases, see Figs. 3(a, b), 4(e, f) and 5(i, j), while the real parts of solutions are in the form of bright solitons and their amplitudes increase as the Ursell parameter increases, see Figs. 4(c, d) and 5(g, h).

Similarly when we use the relation in Eq. (57) with $F^* = -F$, $G^* = -G$, and by means of the same steps, we obtain the following algebraic equations

$$
q a_0^3 - q a_0 b_1^2 = 0, \quad -\Omega a_1 - 2 q a_0 a_1 b_1 = 0, \quad q a_0 a_1^2 + \Omega b_1 + q a_0 b_1^2 = 0, \quad -K^2 p a_1 + q a_0^2 a_1 -3 q a_1 b_1^2 = 0, \quad -2 K^2 p a_1 - q a_1^2 - 3 q a_1 b_1^2 = 0, \quad q a_0^3 b_1 - q b_1^3 = 0, \quad 2 K^2 p b_1 + 3 q a_1^2 b_1 + q b_1^3 = 0.
$$

Solving them gives the following cases

**Figure 4:** The graphs for the elevation of the free surface ($\eta_1$) at different values of Ursell parameter.

**Figure 5:** The graphs for the elevation of the free surface ($u_1$) at different values of Ursell parameter.
Therefore, we have another family of complex exact traveling wave solutions for NLS equation as

\[ u_{11,2} = i K \sqrt{\frac{p}{2q}} \left( -1 \pm \csc h \left( i K (\xi + K p \tau) \right) - \coth \left( i K (\xi + K p \tau) \right) \right), \]

\[ u_{11,4} = i K \sqrt{\frac{p}{2q}} \left( 1 \pm \csc h \left( i K (\xi + K p \tau) \right) - \coth \left( i K (\xi + K p \tau) \right) \right), \]

\[ u_{5,6} = i K \sqrt{\frac{p}{2q}} \left( -1 \pm \csc h \left( i K (\xi - K p \tau) \right) + \coth \left( i K (\xi - K p \tau) \right) \right), \]

\[ u_{7,8} = i K \sqrt{\frac{p}{2q}} \left( 1 \pm \csc h \left( i K (\xi - K p \tau) \right) + \coth \left( i K (\xi - K p \tau) \right) \right), \]

\[ u_{9,10} = \mp i K \sqrt{\frac{2p}{q}} \left( 1 + \coth \left( i K (\xi - 2 K p \tau) \right) \right), \]

\[ u_{11,12} = i K \sqrt{\frac{2p}{q}} \left( \mp 1 \pm \coth \left( i K (\xi + 2 K p \tau) \right) \right). \]

To show the properties of the exact solutions for the NLS equation and the shallow water wave equations, we take some of these solutions as illustrative examples and draw their graphs as Figs. 7, 8, 9 and 10.
Figure 7: The graphs for the solutions of NLS equation \( u_{11} \) at different values of Ursell parameter.

Figure 8: The graphs for the elevation of the free surface \( \eta \) at different values of Ursell parameter.

From the above figures we notice that the real and imaginary parts of all solutions are in the form of solitary traveling waves and their amplitudes increase as the Ursell parameter increases, see Figs. 7(a, b), 8(c, d), 8(e, f), 9(g, h) and 9(i, j).

We see also that the conservation of power \( W \) is satisfied all over the space \( x \) and time \( t \) except at very few spots where it deviates from zero by the order of \( 10^{-14} \) and \( 10^{-18} \) depending on the Ursell parameter, see Figs. 6(k, l) and 10(k, l). This means that the effect of the Ursell parameter plays an important role on the wave profiles.

## 5 Conclusions

The nonlinear water wave problem for an incompressible and inviscid fluid of constant depth is studied by including the effects of the constant gravity acceleration and surface tension. The governing equations of this problem are converted
into shallow water equations by using some asymptotic expansions. We use the reductive perturbation method, which is one of the modern approximation methods, to derive the NLS equation from the nonlinear shallow water wave equations. We also derived the dispersion properties for these equations which are normal since the phase velocity is greater than the group velocity.

Based on the computerizing symbolic computation, in this paper we have presented a complex ansatz method for exact traveling wave solutions to NLS equation. Thus the physical variables of the problem are investigated. The results indicate that the wave profile is significantly influenced by the Ursell parameter. So that depending on this parameter, we draw several figures for the exact solutions, which are in the form of bright and dark solitons. It is found that the wave packet is stable in all the region of $k$-$\delta$ plane except at the very small values.

In this study, the value of Bond number is very small ($7.34 \times 10^{-6}$). This value indicates that the system is relatively unaffected by surface tension effects, and the resulting waves are gravity water waves.

Finally, we observe that the conservation of power is satisfied which means that the suggested methods of solutions
are successfully applied to the tackled problem.

References


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