A Note of the Expected Value and Variance of Fuzzy Variables

Zhigang Wang¹ *, Fanji Tian²

¹ Department of Applied Mathematics, Hainan University, Haikou, 570228, China
² Institute of Mathematics and Computer Science, Hubei University, Wuhan, 430062, China

(Received 15 May 2009, accepted 9 December 2009)

Abstract: The emphasis in this paper is mainly on some properties expected value operator and variance of fuzzy variables, the expected value and variance formulas of three common types of fuzzy variables. In the end, the expected value and variance of geometric Liu process are given.

Keywords: fuzzy variables, expected value, variance; geometric Liu process.

1 Introduction

Probability theory is a branch of mathematics for studying the behavior of random phenomena based on the normalality, nonnegativity, and countable additivity axioms. However, the additivity axiom of classical measure theory has been challenged by many mathematicians. Sugeno¹ generalized classical measure theory to fuzzy measure theory by replacing the additivity axiom with monotonicity and semicontinuity axioms. In order to deal with general uncertainty, self-duality and countable subadditivity are much more important than continuity and semicontinuity. For this reason, Liu² founded an uncertainty theory that is a branch of mathematics based on normality, monotonicity, self-duality and countable subadditivity axioms. Uncertainty provides the commonness of probability theory, credibility theory and chance theory. In fact, probability, credibility, chance, and uncertain measures meet the normality, monotonicity and self-duality axioms. The essential difference among those measures is how to determine the measure of union. Since additivity and maximality are special cases of subadditivity, probability and credibility are special cases of chance measure, and three of them are in the category of uncertain measure. This fact also implies that random variable and fuzzy variable are special cases of hybrid variables, and three of them are instances of uncertain variables. The emphasis in this paper is mainly on some properties expected value operator and variance of fuzzy variables, the expected value and variance formulas of three common types of fuzzy variables. In the end, the expected value and variance of geometric Liu process are given.

2 Text

2.1 Credibility measure

In this section, we review some preliminary concepts within the framework of credibility theory. (3-6)

Definition 1 (Liu[2]) Let Θ be a nonempty set, and P(Θ) the power set of Θ. A set function Cr(,) called a credibility measure if it satisfies:

1. (Normality) Cr(\emptyset) = 1;
2. (Monotonicity) Cr\{A\} \leq Cr\{B\} whenever A \subset B;
3. (Self-Duality) Cr\{A\} + Cr\{A^c\} = 1 for any event A;
4. (Maximality) Cr\{\bigcup A_i\} = \sup Cr\{A_i\} for any events \{A_i\} with \sup Cr\{A_i\} < 0.5.

The triplet (Θ, P(Θ), Cr) is called a credibility space.

The law of contradiction tells us that a proposition cannot be both true and false at the same time, and the law of excluded middle tells us that a proposition is either true or false. Self-duality is in fact a generalization of the law of
contradiction and law of excluded middle. In other words, a mathematical system without self-duality assumption will be inconsistent with the laws. This is the main reason why self-duality axiom is assumed.

**Definition 2** (Liu[2]) A fuzzy variable is a function from a credibility space \((\Theta, \mathcal{P}, Cr)\) to the set of real numbers.

**Definition 3** (Liu[2]) Let \(\xi\) be a fuzzy variable defined on the credibility space \((\Theta, \mathcal{P}, Cr)\). Then its membership function is derived from the credibility measure by

\[
\mu(x) = (2Cr[\xi = x]) \land 1, x \in \mathcal{R}
\] (1)

In practice, a fuzzy variable may be specified by a membership function. In this case, we need a formula to calculate the credibility value of some fuzzy event. The credibility inversion theorem provides this[4].

For fuzzy variables, there are many ways to define an expected value operator. See, for example, Dubois and Prade [7], Heilpern [8], Campos and Gonzalez [9], Gonzalez [10] and Yager [11]. The most general definition of expected value operator of fuzzy variable was given by Liu and Liu [3]. This definition is applicable to both continuous and discrete fuzzy variables.

\[\mu(x) \leq \frac{a}{b} \Rightarrow \mu(x) \geq \frac{a}{b}\]

**2.2 Expected value of fuzzy variables**

**Definition 4** (Liu and Liu [4]) Let \(\xi\) be a fuzzy variable. Then the expected value of \(\xi\) is defined by

\[
E[\xi] = \int_0^\infty Cr[\xi \geq r]dr - \int_{-\infty}^0 Cr[\xi \leq r]dr
\] (2)

provided that at least one of the two integrals is finite.

The equipossible fuzzy variable on \([a, b]\) has an expected value \(\frac{a+b}{2}\). The triangular fuzzy variable \((a, b, c)\) has an expected value \(\frac{(a+2b+c)}{4}\). The trapezoidal fuzzy variable \((a, b, c, d)\) has an expected value \(\frac{(a+b+c+d)}{4}\).

Let \(\xi\) be a continuous nonnegative fuzzy variable with membership function \(\mu\). If \(\mu\) is decreasing on \([0, \infty)\), then \(Cr[\xi \geq x] = \frac{\mu(x)}{2}\) for any \(x > 0\), and

\[
E[\xi] = \frac{1}{2} \int_0^\infty \mu(x)dx
\] (3)

**Example 1** A fuzzy variable \(\xi\) is called exponentially distributed if it has an exponential membership function

\[
\mu(x) = 2(1 + \exp(-\frac{\pi x}{\sqrt{6m}}))^{-1}, x \geq 0, m > 0
\] (4)

then the expected value is \(\sqrt{2m} \ln 2\).

**Theorem 1** Let \(\xi\) be a continuous fuzzy variable with membership function \(\mu\). If its expected value exists, and there is a point \(x_0\) such that \(\mu(x)\) is increasing on \((-\infty, x_0)\) and decreasing on \((x_0, +\infty)\), then

\[
E[\xi] = x_0 + \frac{1}{2} \int_{x_0}^{+\infty} \mu(x)dx - \frac{1}{2} \int_{-\infty}^{x_0} \mu(x)dx
\] (5)

**Proof** If \(x_0 \geq 0\), then

\[
Cr[\xi \geq r] = \left\{ \begin{array}{ll} \frac{1}{2}[1 + 1 - \mu(x)], & \text{if } 0 \leq r \leq x_0, \\ \frac{1}{2} \mu(x), & \text{if } r > x_0. \end{array} \right.
\]

and \(Cr[\xi \leq r] = \frac{1}{2} \mu(x)\),

\[
E[\xi] = \int_0^{x_0} [1 - \frac{1}{2} \mu(x)]dx + \int_{x_0}^{+\infty} \frac{1}{2} \mu(x)dx - \int_{-\infty}^{0} \frac{1}{2} \mu(x)dx
\]

\[
= x_0 + \frac{1}{2} \int_{x_0}^{+\infty} \mu(x)dx - \frac{1}{2} \int_{-\infty}^{x_0} \mu(x)dx
\]

If \(x_0 < 0\), a similar way may prove the equation (5). The theorem 1 is proved.

**Example 2** Let \(\xi\) be triangular fuzzy variable \((a, b, c)\), then it has an expected value

\[
E[\xi] = b + \frac{1}{2} \int_{b}^{c} \frac{x - c}{b - c} dx - \frac{1}{2} \int_{a}^{b} \frac{x - a}{b - a} dx = b + \frac{c - b}{4} + \frac{a - b}{4} = \frac{a + 2b + c}{4}
\]
Example 3 Let $\xi$ be equipossible fuzzy variable on $[a, b]$ then it has an expected value
\[
E[\xi] = \frac{a + b}{2} + \frac{1}{2} \int_{a}^{b} 1dx = \frac{a + b}{2}
\]

Example 4 Let $\xi$ be trapezoidal fuzzy variable $(a, b, c, d)$ then it has an expected value
\[
E[\xi] = \frac{b + c}{2} + \frac{1}{2} \int_{a}^{c} 1dx + \frac{1}{2} \int_{c}^{d} \frac{x - d}{c - d} dx + \frac{1}{2} \int_{a}^{b} \frac{x - a}{b - a} dx = \frac{a + b + c + d}{4}
\]

Especially, let $\xi$ be triangular fuzzy variable $(a, b, c)$ and $b - a = c - b$ then it has an expected value $E[\xi] = b$.

Let $\xi$ be trapezoidal fuzzy variable $(a, b, c, d)$ and $b - a = d - c$ then it has an expected value $E[\xi] = \frac{b + c}{2}$.

A fuzzy variable $\xi$ is called normally distributed if it has a normal membership function
\[
\mu(x) = 2(1 + \exp(-\frac{\pi |x - c|}{\sqrt{6}\sigma})))^{-1}, \quad x \in \mathbb{R}, \quad \sigma > 0
\] (6)

By theorem 1, the expected value is $e$.

The definition of expected value operator is also applicable to discrete case. Assume that $\xi$ be a simple fuzzy variable whose membership function is given by
\[
\mu(x) = \begin{cases} 
\mu_1, & \text{if } x = a_1 \\
\mu_2, & \text{if } x = a_2 \\
\ldots \\
\mu_m, & \text{if } x = a_m 
\end{cases}
\]

where $a_1, a_2, \ldots, a_m$ are distinct numbers. Note that $\mu_1 \vee \mu_2 \vee \ldots \vee \mu_m = 1$. Then the expected value of $\xi$ is
\[
E[\xi] = \sum_{i=1}^{m} \omega_i a_i,
\]

where the weights are given by
\[
\omega_i = \frac{1}{2} \left( \max_{1 \leq j \leq m} \{ \mu_j | a_j \leq a_i \} - \max_{1 \leq j \leq m} \{ \mu_j | a_j < a_i \} + \max_{1 \leq j \leq m} \{ \mu_j | a_j \geq a_i \} - \max_{1 \leq j \leq m} \{ \mu_j | a_j > a_i \} \right)
\]

for $i = 1, 2, \ldots, m$. It is easy to verify that all $\omega_i \geq 0$ and the sum of all weights is just 1.

2.3 Some formulas variance of fuzzy variables

The variance of a fuzzy variable provides a measure of the spread of the distribution around its expected value. A small value of variance indicates that the fuzzy variable is tightly concentrated around its expected value; and a large value of variance indicates that the fuzzy variable has a wide spread around its expected value.

Definition 5 (Liu and Liu [4]) Let $\xi$ be a fuzzy variable with finite expected value $e$. Then the variance of $\xi$ is defined by
\[
V[\xi] = E[(\xi - e)^2].
\]

Definition 6 (Liu [12]) Let $\xi$ be a fuzzy variable, and $k$ a positive number. Then the expected value $E[\xi^k]$ is called the $k$th moment.

Theorem 2 Let $\xi$ be a normal fuzzy variable with membership function (6). Then the variance is $\sigma^2$.

Proof \[ E[\xi] = e, \]
\[
Cr\{(|\xi - e|) ^2 \geq r\} = Cr\{(|\xi - e|) \geq \sqrt{r}\} \cup \{(\xi - e) \leq -\sqrt{r}\}
\] = \[
Cr\{(|\xi - e|) \geq \sqrt{r}\} \vee Cr\{(|\xi - e|) \leq -\sqrt{r}\}
\] = \[
Cr\{(|\xi - e|) \geq \sqrt{r}\}
\] = \[
(1 + \exp(\frac{\pi}{\sqrt{6}\sigma}))^{-1}
\]

then the variance
\[
V[\xi] = \int_{e}^{\infty} (1 + \exp(\frac{\pi}{\sqrt{6}\sigma}))^{-1} dr = \sigma^2
\]
Let $\xi$ be a exponential fuzzy variable with membership function (4). Then the second moment is $m^2$.

**Theorem 4** Let $\xi$ be a triangular fuzzy variable $(a,b,c)$, then its variance

$$V[\xi] = \begin{cases} \frac{33\alpha^3 + 11\alpha^2 \beta + 21\alpha \beta^2 - \beta^3}{384\alpha}, & \alpha > \beta \\ \frac{\alpha^2}{2\alpha}, & \alpha = \beta \\ \frac{33\beta^3 + 11\beta^2 \alpha + 21\beta \alpha^2 - \alpha^3}{384\beta}, & \alpha < \beta \end{cases} \tag{7}$$

where $a = b - a, \beta = c - b$.

**Proof** Let $m = E[\xi]$, when $\alpha = \beta$, then $m = b$ and

$$Cr\{(\xi - m)^2 \geq r\} = \begin{cases} \frac{a - \sqrt{r}}{2\alpha}, & 0 \leq r \leq \alpha^2 \\ \frac{\alpha - \sqrt{r}}{\alpha}, & r \geq \alpha^2 \end{cases}$$

then

$$V[\xi] = E[(\xi - m)^2] = \int_0^\infty Cr\{(\xi - m)^2 \geq 2\} dr = \frac{\alpha^2}{6}$$

when $\alpha > \beta, m < b$

$$Cr\{(\xi - m)^2 \geq r\} = Cr\{(\xi - m) \geq \sqrt{r}\} \lor Cr\{(\xi - m) \leq -\sqrt{r}\}$$

(i) $0 \leq r \leq (b - m)^2$, then

$$Cr\{(\xi - m)^2 \geq r\} = Cr\{(\xi - m) \geq \sqrt{r}\} = \frac{1}{2} \left[ 1 + \frac{\alpha - \sqrt{r} - \sqrt{r + m - a}}{2\alpha} \right] = 1 - \frac{\sqrt{r + m - b} + \alpha}{2\alpha}$$

(ii) when $(b - m)^2 \leq r \leq r_s$, where $r_s = \frac{(\alpha + \beta)^2}{16}$, then

$$Cr\{(\xi - m)^2 \geq r\} = Cr\{(\xi - m) \geq \sqrt{r}\} = \frac{1}{2} \left[ \frac{m - \beta - \sqrt{r} + \alpha}{2\alpha} \right] = \frac{m - \sqrt{r + (b + \beta)}}{2\beta}$$

(iii) when $r_s \leq r \leq (b - m - 3\alpha)^2$, then

$$Cr\{(\xi - m)^2 \geq r\} = Cr\{(\xi - m) \leq -\sqrt{r}\} = \frac{1}{2} \left[ \frac{m - \sqrt{r} - \alpha}{2\alpha} \right] = \frac{m - \sqrt{r} + \alpha}{2\alpha}$$

(iv) when $r \geq (b - m - 3\alpha)^2$, $Cr\{(\xi - m)^2 \geq r\} = 0$ then,

$$V[\xi] = E[\xi^2] = \int_0^\infty Cr\{(\xi - m)^2 \geq 2\} dr = \frac{33\alpha^3 + 11\alpha^2 \beta + 21\alpha \beta^2 - \beta^3}{384\alpha}$$

when $\alpha > \beta, a$ similar way may prove the equation (7). The theorem 4 is proved.

**Theorem 5** ([Chen 13]) Let $\xi$ be a trapezoidal fuzzy variable $(a,b,c,d)$, then its variance

(i) when $\alpha = \beta$,

$$V[\xi] = \frac{3(b - a + \beta)^2 + \beta^2}{24}$$

(ii) when $\alpha > \beta$,

$$V[\xi] = \begin{cases} \frac{1}{6} \left[ \frac{(b - m)^3}{\beta} \right] - \frac{(a - \alpha - m)^3}{\alpha} + \frac{(b\alpha a - m(\alpha + \beta))^3}{\alpha^2(\alpha - \beta)^2}, & a - m < 0 \\ \frac{1}{6} \left[ \frac{(a - m)^3}{\alpha} \right] - \frac{(b - m)^3}{\beta} - \frac{(a - \alpha - m)^3}{\alpha} + \frac{(b\alpha a - m(\alpha + \beta))^3}{\alpha^2(\alpha - \beta)^2}, & a - m \geq 0 \end{cases}$$

(iii) when $\alpha < \beta$,

$$V[\xi] = \begin{cases} \frac{1}{6} \left[ \frac{(a - m)^3}{\alpha} \right] + \frac{(b + \beta - m)^3}{\beta} - \frac{(c\beta - m(\alpha + \beta))^3}{\alpha^2(\alpha - \beta)^2}, & b - m > 0 \\ \frac{1}{6} \left[ \frac{(a - m)^3}{\alpha} \right] + \frac{(b - m)^3}{\beta} - \frac{(c\beta - m(\alpha + \beta))^3}{\alpha^2(\alpha - \beta)^2}, & b - m \leq 0 \end{cases}$$

IJNS homepage: http://www.nonlinearscience.org.uk/
**Theorem 6** Let $\xi$ be a triangular fuzzy variable $(a,b,c)$. Then $V[\xi]$ is an increasing function of $\alpha$ and $\beta$, where $\alpha = b - a, \beta = c - b$.

**Proof** (i) when $\alpha = \beta$, the equation holds trivially.
(ii) when $\alpha > \beta$, set $\alpha' < \alpha$, then
\[
\frac{33\alpha_1^3 + 11\alpha_1\beta^2 + 21\alpha_1^2\beta - \beta^3}{384\alpha_1} - \frac{33\alpha_2^3 + 11\alpha_2\beta^2 + 21\alpha_2^2\beta - \beta^3}{384\alpha_2} < 0
\]
$V[\xi]$ is an increasing function of $\alpha$.

On the other hand,
\[
\left(\frac{33\alpha^3 + 11\alpha\beta^2 + 21\alpha^2\beta - \beta^3}{384\alpha}\right)' = \frac{22\alpha\beta + 21\alpha^2 - 3\beta^2}{384\alpha} > 0
\]
$V[\xi]$ is an increasing function of $\beta$.
(iii) A similar way may prove the result. The theorem 6 is proved.

**Theorem 7** Let $\xi$ be a trapezoidal fuzzy variable $(a,b,c,d)$. Then $V[\xi]$ is an increasing function of $\alpha$ and $\beta$. It is also an increasing function of $c - b$, where $\alpha = b - a, \beta = d - c$.

**Proof** (1) when $\alpha = \beta$, the equation holds trivially.
(2) where $\alpha > \beta$, let $h = b - a$, then
\[
a - m = \frac{1}{4}(\alpha - \beta - 2h), b - m = \frac{1}{4}(\alpha - \beta + 2h),
\]
\[
a - \alpha - m = \frac{1}{4}(3\alpha + \beta + 2h), \beta a + \alpha b - m(\alpha + \beta) = \frac{1}{4}(\alpha + \beta + 2h)(\alpha - \beta)
\]
when $a - m < 0$
\[
V[\xi] = \frac{1}{32}[3(2h + \alpha)^2 + \beta^2 + (3\alpha + \beta + 2h)^2 + (\alpha + \beta + 2h)^2 + (3\alpha + \beta + 2h)(\alpha + \beta + 2h)]
\]
when $a - m \geq 0$,
\[
V[\xi] = \frac{1}{64\alpha}(60\alpha h^2 + 66\alpha^2 h + 33\alpha^3 + 11\alpha\beta^2 + 21\alpha^2\beta + 36\alpha\beta h - 8h^3 - 6\beta^2 h - 12\beta h^2 - \beta^3)
\]
It is easy to prove that it is an increasing function of $\alpha, \beta$ and $h$, respectively.
(3) when $\alpha < \beta$, a similar way may prove the result. The theorem 7 is proved.

### 2.4 Expected value and variance of Geometric Liu process

**Definition 7** ([Liu 14]) A fuzzy process $C_t$ is said to be a Liu process if
(i) $C_0 = 0$.
(ii) $C_t$ has stationary and independent increments.
(iii) every increment $C_{t+1} - C_t$ is a normally distributed fuzzy variable with expected value $et$ and variance $\sigma^2 t^2$.
The parameters $e$ and $\sigma^2$ are called the drift and diffusion coefficients, respectively. Liu process is said to be standard if $e = 0$ and $\sigma = 1$.

**Definition 8** ([Liu 14]) Let $C_t$ be a standard Liu process. Then the fuzzy process $G_t = \exp(\alpha t + \sigma C_t)$ is called a geometric Liu process.

In 2008, Li [15] proved that $G_t$ has a lognormal membership function and obtained its expected value and variance. In addition, alternative fuzzy stock models were introduced by Peng [16].

**Theorem 8** Let $G_t$ be a geometric Liu process whose membership function
\[
\mu(x) = 2(1 + \exp\left(\frac{\pi|\ln x - et|}{\sqrt{6}\sigma t}\right))^{-1}, x > 0
\]
then it has expected value $E[G_t] = \exp(\alpha t) \csc(\sqrt{6}\sigma t)\sqrt{6}\sigma t, \forall t < \pi(\sqrt{6}\sigma)^{-1}$.
Proof\ ∀t < \pi(\sqrt{6}\sigma)^{-1}

\[ E[G_t] = \int_0^\infty C_t(r)\{G_t \geq r\}dr = \int_0^{e^{et}} \left[1 - \left(1 + \exp\left(\frac{\pi(t - \ln(x))}{\sqrt{6}\sigma}\right)\right)^{-1}\right]dx + \int_{e^{et}}^\infty \left[1 + \exp\left(\frac{\pi(t - \ln(x))}{\sqrt{6}\sigma}\right)\right]^{-1}dx \]

\[ = \int_0^\infty \exp\left(\frac{\pi(t - \ln(x))}{\sqrt{6}\sigma}\right)\frac{dx}{x} + \int_{e^{et}}^\infty \exp\left(\frac{\pi(t - \ln(x))}{\sqrt{6}\sigma}\right)\frac{dx}{x} \]

\[ = \exp(et) \int_0^\infty \left(1 + \frac{x}{\sqrt{6}\sigma}\right)^{-1}dx = \exp(et) \csc(\sqrt{6}\sigma t)\sqrt{6}\sigma t \]

\[ V[G_t] = \int_0^E[G_t] - \exp(2et) \left(1 + \exp\left(\frac{\pi(t - \ln(x))}{\sqrt{6}\sigma}\right)\right)^{-1}dx + \int_{E[G_t]}^\infty \left[1 + \exp\left(\frac{\pi(t - \ln(x))}{\sqrt{6}\sigma}\right)\right]^{-1}dx \]

 Otherwise, \( V[G_t] = +\infty. \)

3 Conclusions

The main contribution of the present paper is to obtain the expected value and variance formulas of the triangular fuzzy variable, the trapezoidal fuzzy variable, normal fuzzy variable and geometric Liu process. On the other hand, some properties of variance are given.

Acknowledgments

This work is supported by the institution of higher learning scientific research Program (No.Hjkj200908) of Hainan Provincial Department of Education, the Hainan Natural Science Foundation(No.807025),China.

References

