Exact Solutions for Non-linear Volterra-Fredholm Integro-Differential Equations by He’s Homotopy Perturbation Method

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Abstract: In this article, an application of He’s homotopy perturbation method is applied to solve non-linear Volterra-Fredholm integro-differential equations. Some non-linear examples are prepared to illustrate the efficiency and simplicity of the method.

Keywords: homotopy perturbation method; non-linear Volterra-Fredholm integro-differential equations

1 Introduction

Homotopy perturbation method has been used by many mathematicians and engineers to solve various functional equations. This method continuously deforms difficult problem, mostly because of non-linearly, into a simple, linear, equation [1-5]. Almost all perturbation methods are based on the assumption of the existence of a small parameter in the equation. But most non-linear problems do not have such a small parameter. This method has been proposed to eliminate the small parameter [6, 7]. In recent years, the application of homotopy perturbation theory has appeared in many researches [10-14]. In this paper, we propose homotopy perturbation method to solve non-linear Volterra-Fredholm integro-differential equations. Consider the following equation [8]

\[ \sum_{j=0}^{m} p_j(x)y^j(x) = f(x) + \lambda_1 \int_{a}^{x} k_1(x, t) [y(t)]^p dt + \lambda_2 \int_{a}^{b} k_2(x, t) [y(t)]^q dt, \]  

(1)

where \( p_j(x) (j = 0 \ldots m) \), \( f(x), k_1(x, t), k_2(x, t) \) are function having \( n \) th \( (n \geq m) \) derivatives on an integral \( a \leq x, t \leq b \), and \( a, b, \lambda_1 \) and \( \lambda_2 \) are constants, \( p \) and \( q \) are positive integer.

2 Homotopy perturbation method

To illustrate the homotopy perturbation method, we consider (1) as

\[ L(y) = \sum_{j=0}^{m} p_j(x)y^j(x) - f(x) - \lambda_1 \int_{a}^{x} k_1(x, t) [y(t)]^p dt - \lambda_2 \int_{a}^{b} k_2(x, t) [y(t)]^q dt, \]  

(2)

with boundary conditions,

\[ B(y, \partial y/\partial n) = 0, \]

and exact solution \( y(x) = g(x) \). By the homotopy technique, we can define homotopy \( H(y, p) \) by

\[ H(y, p) = (1 - p)F(y) + pL(y), \]  

(3)

where \( F(y) \) is a functional operator with known solution \( y_0 \) which generally satisfies the boundary conditions. Obviously, from Eq. (3) we have

\[ H(y, 0) = F(y), \quad H(y, 1) = L(y), \]

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and the changing process of the parameter \( p \) from 0 to 1, is just that of \( H(y, p) \) from \( H(y, 0) \) to \( H(y, 1) \). In topology, it is called deformation, \( H(y, 0) \) and \( H(y, 1) \) are called homotopic. Applying the perturbation technique \([10]\), due to the fact that \( 0 \leq p \leq 1 \), can be considered as a small parameter. We can assume that the solution of (3) can be expressed as a series in as follows:

\[
y = y_0 + py_1 + p^2 y_2 + \cdots.
\]  

(4)

As \( p \to 1 \), (3) tends to equation (2) and (4), in most cases it converges to an approximate solution of (2), i.e.,

\[
g = \lim_{p \to 1} y = y_0 + y_1 + y_2 + \cdots.
\]

3 Numerical Examples

In this part, three examples are provided. These examples are considered to illustrate the ability and reliability of the method. These examples are solved numerically in \([15, 16]\).

**Example 1.** Let’s solve the following non-linear Volterra-Fredholm integro-differential equation \([15]\)

\[
y'(x) + 2xy(x) = f(x) + \int_0^x (x + t)[y(t)]^3 dt + \int_0^1 (x - t)y(t)dt,
\]

where \( f(x) = (-\frac{2}{3}x + \frac{1}{3})e^{x} + (2x + 1)e^{x} + (\frac{4}{3} - e)x + \frac{8}{3} \), with condition \( y(0) = 1 \) and exact solution \( y(x) = e^x \).

We construct a homotopy \( \Omega \times [0, 1] \to \mathbb{R} \) which satisfies

\[
y'(x) + (1 - p)2xe^x + p2xy(x) - 
\]

\[
f(x) - \int_0^1 (1 - p)(x + t)[e^t]^3 + p(x + t)[y(t)]^3 dt - \int_0^1 (1 - p)(x - t)e^t + p(x - t)y(t)dt = 0.
\]  

(5)

Substituting (4) in to (5), and equating the coefficients of the terms with the identical powers of \( p \), we have:

\[
p^0 : y_0'(x) + 2xe^x = f(x) + \int_0^x (x + t)[e^t]^3 dt + \int_0^1 (x - t)e^t dt = 0 \Rightarrow y_0(x) = e^x.
\]

\[
p^1 : y_1'(x) - 2xe^x + 2xy_0(x) = \int_0^x -(x + t)[e^t]^3 + (x + t)[y_0(t)]^3 dt 
\]

\[
+ \int_0^1 -(x - t)e^t + (x - t)y_0(t)dt = 0 \Rightarrow y_1(x) = 0.
\]

\[
p^2 : y_2'(x) + 2xy_1(x) = \int_0^x (x + t)[y_1(t)]^3 dt + \int_0^1 (x - t)y_1(t)dt = 0 \Rightarrow y_2(x) = 0.
\]

\[
p^3 : y_3'(x) + 2xy_2(x) = \int_0^x (x + t)[y_2(t)]^3 dt + \int_0^1 (x - t)y_2(t)dt = 0 \Rightarrow y_3(x) = 0.
\]

And by repeating this approach, we obtain

\[
y_4(x) = y_5(x) = \cdots = 0.
\]

Therefore, the approximation to the solution of Example 1 can be readily obtained by

\[
y = \sum_{i=0}^{\infty} y_i = e^x + 0 + 0 + \cdots.
\]

And hence,

\[
y(x) = e^x,
\]

which is an exact solution of Example 1.

**Example 2.** Consider the following Volterra-Fredholm integro-differential equation \([15]\)

\[
y''(x) - xy'(x) + xy(x) = f(x) + \int_{-1}^{x} (x - 2t)[y(t)]^2 dt + \int_{-1}^{1} xy(t)dt,
\]
where \( f(x) = \frac{2}{27} x^6 - \frac{1}{6} x^4 + x^3 - 2 x^2 - \frac{23}{15} x + \frac{5}{3}, \) with condition \( y(0) = -1, y'(0) = 0 \) and exact solution \( y(x) = x^2 - 1. \)

By using the He’s homotopy perturbation method, we have

\[
y''(x) - (1 - p)x(2x) - px'y'(x) + (1 - p)x(x^2 - 1) + px y(x) - f(x)
- \int_{-1}^{x} (1 - p)(x - 2t)[x^2 - 1]^2 + p(x - 2t)[y(t)]^2 dt + \int_{-1}^{1} (1 - p)xt(x^2 - 1) + pxtg(t)dt = 0. \tag{6}
\]

Substituting (4) in to (6), and equating the coefficients of the terms with the identical powers of \( p, \) we have:

\[
p^0 : y''_0(x) - x(2x) + x(x^2 - 1) - f(x) - \int_{-1}^{1} (x - 2t)[x^2 - 1]^2 dt \\
+ \int_{-1}^{x} xt(x^2 - 1)dt = 0, \Rightarrow y_0(x) = x^2 - 1,
\]

\[
p^1 : y''_1(x) + x(2x) - xy'_0(x) - x(x^2 - 1) + xy_0(x)
- \int_{-1}^{x} -(x - 2t)[x^2 - 1]^2 + (x - 2t)[y_0(t)]^2 dt + \int_{-1}^{1} -xt(x^2 - 1) + xty_0(t)dt = 0, \Rightarrow y_1(x) = 0,
\]

\[
p^2 : y''_2(x) - xy'_1(x) + xy_1(x) - \int_{-1}^{x} (x - 2t)[y_1(t)]^2 dt + \int_{-1}^{1} xty_1(t)dt = 0 \Rightarrow y_2(x) = 0,
\]

\[
p^3 : y''_3(x) - xy'_2(x) + xy_2(x) - \int_{-1}^{x} (x - 2t)[y_2(t)]^2 dt + \int_{-1}^{1} xty_2(t)dt = 0 \Rightarrow y_3(x) = 0.
\]

And by repeating this approach, we obtain

\[ y_4(x) = y_5(x) = ... = 0. \]

Therefore the approximation to the solution of Example 2 can be readily obtained by

\[ y = \sum_{i=0}^{\infty} y_i = x^2 - 1 + 0 + 0 + \cdots. \]

And hence,

\[ y(x) = x^2 - 1, \]

which is an exact solution of Example 2.

**Example 3.** Consider the following non-linear Volterra-Fredholm integro-differential equation [16]

\[
y''(x) + y(x) = f(x) + \int_{0}^{x} y^2(t)dt + \int_{0}^{1} (x^2t + xt^2)dt,
\]

where \( f(x) = -\frac{1}{5} x^5 + \frac{2}{3} x^3 + \frac{5}{6} x^2 - \frac{113}{108} x - 1, \) with condition \( y(0) = 1, y'(0) = 0, y''(0) = -2, \) and exact solution \( y(x) = 1 - x^2. \)

He’s homotopy perturbation method consists of the following scheme

\[
y''(x) + (1 - p)(1 - x^2) + py(x) - f(x) - \int_{0}^{x} (1 - p)[1 - x^2]^2 + p[y(t)]^2 dt \\
- \int_{0}^{1} (1 - p)(x^2t + xt^2)[1 - x^2]^2 + p(x^2t + xt^2)[y(t)]^2 dt = 0. \tag{7}
\]

Substituting (4) in to (7), and equating the coefficients of the terms with the identical powers of \( p, \) we have:
accurate solutions were derived from first-order approximations in the examples presented in this paper. The homotopy perturbation method is valid for not only weakly non-linear equations but also for strong ones. Furthermore, the results have been approved the efficiency of this method for solving these problems. The solution obtained by In this work, we used homotopy perturbation method for solving non-linear Volterra-Fredholm integro-differential equations. And by repeating this approach, we obtain

And hence, \( y(x) = 1 - x^2 \), which is an exact solution of Example 3.

\[ y(x) = \sum_{i=0}^{\infty} y_i = 1 - x^2 + 0 + 0 + \cdots. \]

Therefore the approximation to the solution of Example 3 can be readily obtained by

4 Conclusion

In this work, we used homotopy perturbation method for solving non-linear Volterra-Fredholm integro-differential equations. The results have been approved the efficiency of this method for solving these problems. The solution obtained by homotopy perturbation method is valid for not only weakly non-linear equations but also for strong ones. Furthermore, accurate solutions were derived from first-order approximations in the examples presented in this paper.

References


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