A Series of Traveling Wave Solutions for Nonlinear Evolution Equations Arising in Physics

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Abstract: An extended algebraic method is devised to uniformly construct a series of complete exact solutions for some nonlinear evolution equations arising in physics. For illustration, we apply the proposed method to three nonlinear coupled physical systems, namely, generalized Hirota-Satsuma coupled KdV system, coupled Maccari equations, and generalized Zakharov equations. The solutions obtained include soliton solutions, rational solutions, triangular periodic solutions, Jacobi and Weierstrass doubly periodic wave solutions. Among them, the Jacobi elliptic periodic wave solutions exactly degenerate to the soliton solutions at a certain limit condition. It is shown that the extended algebraic method provides a very effective and powerful mathematical tool for solving other nonlinear evolution equations arising in physics.

Keywords: extended algebraic method; nonlinear evolution equations; new exact wave solutions

1 Introduction

Mathematical modelling of physical systems often leads to nonlinear evolution equations. Explicit solutions (or more precisely class of solutions) to such equations are of fundamental importance. In particular, there is considerable interest in explicit traveling waves or solitary wave solutions. Nonlinear waves are encountered in numerous domains such as fluid mechanics, solid state physics, plasma physics, and chemical physics. Names like solitons, kinks, breathers, etc. are commonly used in the vast literature dealing with this subject. A key step in investigating nonlinear problems is the derivation of traveling waves from the associated evolution equations. However, in contrast with linear wave theory where one can make use of the basic technique of Fourier analysis, we are here confronted with a large variety of methods. Several standard methods for obtaining such solutions are known [1 – 20].

In this paper, we present an effective extension to the tanh method and develop a new extended algebraic method to uniformly construct a series of travelling wave solutions including soliton, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions for general nonlinear equations. For illustration, we apply the proposed method to three nonlinear physical models, namely, generalized Hirota-Satsuma, coupled Maccari system, and generalized Zakharov equations.

2 The proposed method

For a given nonlinear evolution equation

\[ H(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, ...) = 0 \]  

(1)

The main steps of our proposed method are given as follows:

Step 1. By using the travelling wave transformation \( u(x, t) = u(\xi) \), \( \xi = x - ct + l \), then Eq.(1) reduces to an ordinary differential equations

\[ H(u, u', u'', u''', ...) = 0 \]  

(2)

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Step 2. We introduce a new variable $\phi = \phi(\xi)$ which is a solution of the following first order ODE

$$
\phi' = \left[ \sum_{j=0}^{r} h_j \phi^j \right]
$$

Then the derivatives with respect to the variable $\xi$ become the derivatives with respect to the variable $\phi$ as

$$
\frac{d}{d\xi} = \left[ \sum_{j=0}^{r} h_j \phi^j \right] \frac{d}{d\phi}, \quad \frac{d^2}{d\xi^2} = \frac{1}{2} \sum_{j=1}^{r} j h_j \phi^{j-1} \frac{d}{d\phi} + \sum_{j=0}^{r} j h_j \phi^j \frac{d^2}{d\phi^2}
$$

Step 3. The next crucial step is that the solution we are looking for is expressed in the form

$$
u(\xi) = a_0 + \sum_{i=1}^{n} [a_i \phi^i(\xi) + b_i \phi^{-i}(\xi) + c_i \phi^{i-1}(\xi) \phi'(\xi) + d_i \phi^{-i}(\xi) \phi'(\xi)],
$$

where $a_0, a_i, b_i, c_i, d_i$ are constants to be determined later. Balancing the highest derivative term with nonlinear term in (2), we obtain a relation between the positive integers $n$ and $r$, from which the different possible values of $n$ and $r$ can be obtained. For example, in the case of KdV equation

$$
u_t + 6 \nu \nu_x + \nu_{xxx} = 0,$$

we have

$$r = n + 2
$$

If we take $n = 1$ and $r = 3$ in Eq.(6), we may use the following series expansion as a solution of KdV equation

$$
u(\xi) = a_0 + a_1 \phi(\xi) + b_1 \phi^{-1}(\xi) + c_1 \phi'(\xi) + d_1 \phi^{-1}(\xi) \phi'(\xi),
$$

Similarly, if we take $n = 2, r = 4$ in Eq.(6), we have

$$
u(\xi) = a_0 + a_1 \phi(\xi) + a_2 \phi^{2}(\xi) + b_1 \phi^{-1}(\xi) + b_2 \phi^{-2}(\xi) + c_1 \phi'(\xi) + c_2 \phi'(\xi) \phi(\xi) + d_1 \phi^{-1}(\xi) \phi'(\xi) + d_2 \phi^{-2}(\xi) \phi'(\xi),
$$

Step 4. Inserting (3) into (2) and then setting the coefficients of all powers of $\phi^i$ and $\phi' \sqrt{\sum_{j=0}^{r} h_j \phi^{j}(i = 0, 1, ..., n)} (j = 0, 1, ..., r)$ to zero, we will get a system of algebraic equations, from which the constants $a_0, a_1, b_1, c_1, d_1$ and $\xi$ can be found explicitly. Step 5. Substituting the values $a_0, a_1, b_1, c_1, d_1$ obtained in Step 4 back into Eq.(5) and then solving it, we may get all its possible solutions. In this paper, we only consider the case $r = 4$ and hence

$$
\phi'(\xi) = \sqrt{h_0 + h_1 \phi(\xi) + h_2 \phi(\xi)^2 + h_3 \phi(\xi)^3 + h_4 \phi(\xi)^4}
$$

By considering different values of $h_0, h_1, h_2, h_3$ and $h_4$, we have the following results:

Case A. Eq.(7) admits two kinds of polynomials solutions as follows

$$
\phi_1 = \sqrt{h_0} \xi, \quad h_1 = h_2 = h_3 = h_4 = 0, \quad h_0 > 0
$$

$$
\phi_2 = -\frac{h_0}{h_1} + \frac{1}{4} h_1 \xi^2, \quad h_2 = h_3 = h_4 = 0, \quad h_1 \neq 0
$$

Case B. Eq.(7) admits two kinds of exponential solutions, namely,

$$
\phi_3 = \frac{-h_1}{2 h_2} + \exp(\sqrt{h_2} \xi), \quad h_3 = h_4 = 0, \quad h_0 = \frac{h_1^2}{4 h_2}, \quad h_2 > 0
$$

$$
\phi_4 = \frac{h_3}{2 h_4} \exp(\frac{h_3}{2 \sqrt{-h_4}} \xi), \quad h_0 = h_1 = h_2 = 0, \quad h_4 > 0
$$
Case C. Eq. (7) admits two kinds of rationals, namely,

\[
\phi_5 = -\frac{1}{\sqrt{h_4}}, h_0 = h_1 = h_2 = h_3 = 0, h_4 > 0
\]

\[
\phi_6 = -\frac{4h_3}{h_4^2 - 4h_4}, h_0 = h_1 = h_2 = 0
\]

Case D. Eq. (7) admits the triangular solution as follows

\[
\phi_7 = -\frac{h_1}{2h_2} + \frac{h_1}{2h_2} \sin(\sqrt{-h_2}\xi), h_0 = h_3 = h_4 = 0, h_2 < 0,
\]

\[
\phi_8 = \sqrt{-\frac{h_0}{h_2}} \sin(\sqrt{-h_2}\xi), h_1 = h_3 = h_4 = 0, h_2 > 0, h_0 < 0,
\]

\[
\phi_9 = \sqrt{-\frac{h_0}{h_4}} \sec(\sqrt{-h_2}\xi), h_0 = h_1 = h_3 = 0, h_2 < 0, h_4 > 0,
\]

\[
\phi_{10} = \frac{h_2}{h_3} \sec^2(\sqrt{-h_2}\xi), h_0 = h_1 = h_4 = 0, h_2 < 0,
\]

\[
\phi_{11} = \sqrt{\frac{h_2}{2h_4}} \tan(\sqrt{h_2/2}\xi), h_1 = h_3 = 0, h_0 = \frac{h_2^2}{4h_4}, h_2 > 0, h_4 > 0
\]

Case E. Eq. (7) admits the kinds of hyperbolic solutions, namely,

\[
\phi_{12} = -\frac{h_1}{2h_2} + \frac{h_1}{2h_2} \sinh(2\sqrt{h_2}\xi), h_0 = h_3 = h_4 = 0, h_2 > 0,
\]

\[
\phi_{13} = \sqrt{\frac{h_0}{h_2}} \sinh(\sqrt{h_2}\xi), h_1 = h_3 = h_4 = 0, h_2 > 0, h_0 > 0,
\]

\[
\phi_{14} = \sqrt{-\frac{h_0}{h_4}} \sinh(\sqrt{h_2}\xi), h_0 = h_1 = h_3 = 0, h_2 > 0, h_4 < 0,
\]

\[
\phi_{15} = -\frac{h_2}{h_4} \sinh^2(\sqrt{-h_2}\xi), h_0 = h_1 = h_4 = 0, h_2 > 0,
\]

\[
\phi_{16} = \sqrt{-\frac{h_2}{2h_4}} \tanh(\sqrt{-h_2/2}\xi), h_1 = h_3 = 0, h_0 = \frac{h_2^2}{4h_4}, h_2 < 0, h_4 > 0
\]

Case F. Eq. (7) admits three Jacobi elliptic function solutions as follows

\[
\phi_{17} = \sqrt{-h_4} \left(\frac{h_2}{(2m^2 - 1)} \right) \cn(\frac{h_2}{(2m^2 - 1)}\xi), h_4 < 0, h_2 > 0, h_1 = h_3 = 0, h_0 = \frac{m^2 h_2^2}{h_4(2m^2 - 1)^2},
\]

\[
\phi_{18} = \sqrt{-h_4} \left(\frac{h_2}{(m^2 + 1)} \right) \sn(\frac{h_2}{(m^2 + 1)}\xi), h_4 > 0, h_2 > 0, h_1 = h_3 = 0, h_0 = \frac{h_2^2 h_4}{h_4(2m^2 + 1)^2},
\]

\[
\phi_{19} = \sqrt{-h_4} \left(\frac{h_2}{(2m^2 - 2)} \right) \dn(\frac{h_2}{(2m^2 - 2)}\xi), h_4 < 0, h_2 > 0, h_0 = \frac{h_2^2 (1-m^2)}{h_4(2m^2 - 2)^2}, h_1 = h_3 = 0
\]

As \( m \to 1 \), the Jacobi elliptic periodic solutions degenerate to hyperbolic functions, i.e. \( \sn(\xi) = \tanh(\xi) \), \( \cn(\xi) = \sech(\xi) \) and \( \dn(\xi) = \sech(\xi) \). When \( m \to 0 \), the Jacobi elliptic periodic solutions degenerate to triangular functions, i.e. \( \sn(\xi) = \sin(\xi) \), \( \cn(\xi) = \cos(\xi) \) and \( \dn(\xi) = 1 \).

Case G. Eq. (7) admits a Weierstrass elliptic function solution

\[
\phi_{20} = \psi(\sqrt{-h_2/2}\xi, g_2, g_3), h_2 = h_4 = 0, h_3 > 0
\]

where \( g_2 = -\frac{4h_4}{g_3} \) and \( g_3 = \frac{4h_4}{g_3} \).

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3 New applications

In this section, we will demonstrate our proposed approach on three nonlinear evolution equations arising in physics, namely, generalized Hirota-Satsuma coupled KdV system, coupled Maccari equations and generalized-Zakharov equations.

3.1 Example(1). Generalized Hirota-Satsuma coupled KdV system

Let us first consider the Hirota-Satsuma coupled KdV system [13]

\[ u_t = \frac{1}{4} u_{xxx} + 3a u_x + 3(w - v^2) x, \quad v_t = -\frac{1}{2} v_{xxx} - 3a v_x, \quad w_t = -\frac{1}{2} w_{xxx} - 3a w_x \tag{29} \]

When \( w = 0 \), Eqs. (29) reduce to be the well-known Hirota-Satsuma coupled KdV system. We seek travelling wave solutions for Eqs. (29) in the form

\[ u(x, t) = u(\xi), \quad v(x, t) = v(\xi), \quad w(x, t) = w(\xi), \quad \xi = k(x - ct) \tag{30} \]

Substituting Eq. (30) into (29) yields an ODE as

\[ -c k u' = \frac{1}{4} k^3 u''' + 3 k u' + 3 k (w - v^2)' \tag{31} \]

\[ -c k v' = -\frac{1}{2} k^3 v''' - 3 k u' \tag{32} \]

\[ -c k w' = -\frac{1}{2} k^3 w''' - 3 k u' \tag{33} \]

Let

\[ u = \alpha v^2 + \beta_0 v + \gamma, \quad w = A_0 v + B_0 \tag{34} \]

where \( \alpha, \gamma, \beta_0, A_0 \) and \( B_0 \) are constants. Inserting Eq. (34) into (32) and (33) and integrating once we know that (33) and (34) give rise to the same equation

\[ k^2 v'' = -2 \alpha v^3 - 3 \beta_0 v^2 + 2(\alpha - 3 \gamma) v + k_1, \tag{35} \]

where \( k_1 \) is an integration constant. Integrating (35) once again we have

\[ k^2 v'^2 = -\alpha v^4 - 2 \beta_0 v^3 + 2(\alpha - 3 \gamma) v^2 + 2 k_1 v + k_2, \tag{36} \]

where \( k_2 \) is an integration constant. By means of Eqs. (34-36) we get

\[ k^2 u'' = 2 \alpha k^2 v'^2 + k^2 (2 \alpha v + \beta_0) v'' = 2 \alpha [-\alpha v^4 - 2 \beta_0 v^3 + 2(\alpha - 3 \gamma) v^2 + 2 k_1 v + k_2] \]

\[ + 2(\alpha - 3 \gamma) v^2 + 2 k_1 v + k_2] + (2 \alpha v + \beta_0)[-2 \alpha v^3 - 3 \beta_0 v^2 + 2(\alpha - 3 \gamma) v + k_1] \tag{37} \]

Integrating (31) once we have

\[ \frac{1}{4} k^2 u'' + \frac{3}{2} v^2 + c u + 3(w - v^2) + k_3 = 0, \tag{38} \]

where \( k_3 \) is an integration constant. Inserting (34) and (37) into (38) gives

\[ 3 \alpha c - 3 \alpha \gamma + \frac{3}{4} \beta_0^2 - 3 = 0, \quad \frac{1}{2} \{ \alpha k_1 + \beta_0 c + \gamma \beta_0 \} + A_0 = 0, \tag{39} \]

\[ \frac{1}{4} (2 \alpha k_2 + \beta_0 k_1) + \frac{3}{2} \gamma^2 + c \gamma + 3 B_0 + k_3 = 0 \]

Let

\[ k_1 = \frac{1}{2 \alpha^2} [\beta_0^3 + 2 \alpha \beta_0 - 6 \alpha \beta_0 \gamma], \quad v(\xi) = a P(\xi) - \frac{\beta_0}{2 \alpha} \tag{40} \]
Therefore from Eq.(39), we have
\[ k^2 P''(\xi) - a \left( \frac{3\beta_0^2}{2\alpha^2} + 2c - 6\gamma \right) P(\xi) + 2\alpha a^3 P^3(\xi) = 0 \] (41)

By virtue of the technique of solution, we assume that the solution of Eq.(41) in the series form is
\[ P(\xi) = a_0 + a_1 \phi(\xi) + b_1 \phi^{-1}(\xi) + c_1 \phi'(\xi) + d_1 \phi^{-1}(\xi) \phi'(\xi), \]
where \( \phi(\xi) \) satisfies Eq.(7), \( a_0, a_1, b_1, c_1 \) and \( d_1 \) are constants to be later determined. Substituting (42) along with (8) into (41) and setting the coefficients of all powers of \( \phi' \) and \( \phi' \sqrt{\sum r=0 h_r \phi^r} \) we will get a system of algebraic equations for \( a_0, a_1, b_1, c_1 \) and \( d_1 \). Solving the system of algebraic equations with the aid of MAPLE, we determine the coefficients as :

Case(1):
\[
c = \frac{12\gamma \alpha + 3\beta_0^2}{4\alpha}, a_0 = 0, h_1 = 0, d_1 = d_1, h_0 = \frac{b_1^2}{d_1^2}, h_4 = \frac{a_1^2}{d_1^2}, \quad h_2 = -\frac{6a_1b_1}{d_1^2}, c_1 = 0, h_3 = 0, b_1 = b_1, a_1 = a_1, k = 2\sqrt{-aa\alpha d_1}\]
(43)

Case(2):
\[
c = \frac{12\gamma \alpha + 3\beta_0^2}{4\alpha}, a_0 = -\frac{k^2 h_1}{4\alpha a^3 b_1}, h_1 = h_1, d_1 = 0, h_0 = -\frac{\alpha a^3 h_1^2}{k^2}, h_4 = -\frac{1}{4096} \frac{k^4 h_1^4}{b_1^6 a^9 \alpha^3}, \quad h_2 = -15 \frac{k^2 h_1^2}{32 \alpha a^3 b_1}, c_1 = 0, h_3 = 3 \frac{k^4 h_1^2}{64 b_1^3 a^6 \alpha^2}, b_1 = b_1, a_1 = 1 \frac{1}{64} \frac{k^4 h_1^2}{a^6 b_1^4}, k = k\]
(44)

Case(3):
\[
c = \frac{12\gamma \alpha + 3\beta_0^2}{4\alpha}, a_0 = -\frac{1}{4} \frac{k^2 h_1}{a b_1 \alpha}, h_1 = h_1, d_1 = 0, h_0 = -\frac{\alpha a^3 b_1^2}{k^2}, h_4 = h_4, \quad h_2 = -\frac{3}{8} \frac{k^2 h_1^2}{a^3 b_1^2}, c_1 = 0, h_3 = \frac{3}{8} \frac{k^2 h_1^2}{a^3 b_1^2}, b_1 = b_1, a_1 = 0, k = k\]
(45)

From Case(1), substituting Eq.(43) into (34) and (40), we have a new exact travelling wave solutions for Eqs.(29) as follows
\[
v(\xi) = a_0 + a_1 \phi(\xi) + b_1 \phi^{-1}(\xi) + \frac{d_1 \phi'(\xi)}{\phi(\xi)} + \frac{\beta_0}{2\alpha},\]
\[u(\xi) = \alpha[a_0 + a_1 \phi(\xi) + b_1 \phi^{-1}(\xi)] - \frac{\beta_0}{2\alpha} + b_0, \quad w(\xi) = \alpha[a_0 + a_1 \phi(\xi) + b_1 \phi^{-1}(\xi)] + \gamma,\]
(46)

Knowing Case (1) with Eqs.(46), admits to a series of travelling wave solutions of Eqs.(29): If \( h_1 = h_2 = h_3 = h_4 = 0, h_2 > 0 \), admits to polynomial solution
\[
v_1(\xi) = a_0 + a_1 \sqrt{h_0} \xi + b_1 \frac{h_0}{\sqrt{h_0}} + \frac{d_1 h_0}{\sqrt{h_0}} \frac{\beta_0}{2\alpha} u_1(\xi) = \alpha[v_1(\xi)]^2 + \beta_0[v_1(\xi)] + \gamma, \quad w_1(\xi) = A_0[v_1(\xi)] + B_0\]
(47)

If \( h_0 = h_3 = h_4 = 0, h_2 < 0 \), admits to a triangular solution
\[
v_2(\xi) = a_0 + a_1 \frac{-h_1}{2h_2} + h_1 \sin(\sqrt{-h_2} \xi) + \frac{b_1}{2h_2} \sin(\sqrt{-h_2} \xi) + \frac{h_1}{2h_2} - \frac{h_1}{2h_2} \sin(\sqrt{-h_2} \xi) - \frac{h_1}{2h_2} \sin(\sqrt{-h_2} \xi) - \frac{\beta_0}{2\alpha},\]
\[u_2(\xi) = \alpha[v_2(\xi)]^2 + \beta_0[v_2(\xi)] + \gamma, \quad w_2(\xi) = A_0[v_2(\xi)] + B_0\]
(48)
If \( h_0 = h_3 = h_1 = 0, h_2 < 0, h_4 > 0 \), admits to a triangular solution
\[
v_3(\xi) = a[a_1\sqrt{-\frac{h_2}{h_4}} \sec(\sqrt{-h_2} \xi) + \frac{b_1}{\sqrt{-h_2}} \sec(\sqrt{-h_2} \xi) + d_1 \sqrt{-\frac{h_2}{h_4}} \sec(\sqrt{-h_2} \xi) \tan(\sqrt{-h_2} \xi) \sqrt{-\frac{h_2}{h_4}}] = \frac{\beta_0}{2\alpha},
\]
\( u_3(\xi) = \alpha[v_3(\xi)]^2 + \beta_0[v_3(\xi)] + \gamma, \quad w_3(\xi) = A_0[v_3(\xi)] + B_0 \)

When \( h_1 = h_3 = h_4 = 0, h_0 > 0, h_2 < 0 \), admits to a triangular solution
\[
v_4(\xi) = a[a_1\sqrt{-\frac{h_0}{h_2}} \sin(\sqrt{-h_2} \xi) + \frac{b_1}{\sqrt{-h_2}} \sin(\sqrt{-h_2} \xi) + d_1 \sqrt{-\frac{h_0}{h_2}} \cos(\sqrt{-h_2} \xi) \sqrt{-\frac{h_0}{h_2}}] = \frac{\beta_0}{2\alpha},
\]
\( u_4(\xi) = \alpha[v_4(\xi)]^2 + \beta_0[v_4(\xi)] + \gamma, \quad w_4(\xi) = A_0[v_4(\xi)] + B_0 \)

When \( h_0 = h_1 = h_3 = 0, h_0 > 0, h_4 < 0 \), admits to a hyperbolic solution
\[
v_5(\xi) = a[a_1\sqrt{-\frac{h_2}{h_4}} \mathrm{sech}(\sqrt{-h_2} \xi) + \frac{b_1}{\sqrt{-h_2}} \mathrm{sech}(\sqrt{-h_2} \xi) + d_1 \sqrt{-\frac{h_2}{h_4}} \mathrm{sech}(\sqrt{-h_2} \xi) \tanh(\sqrt{-h_2} \xi) \sqrt{-\frac{h_2}{h_4}}] = \frac{\beta_0}{2\alpha},
\]
\( u_5(\xi) = \alpha[v_5(\xi)]^2 + \beta_0[v_5(\xi)] + \gamma, \quad w_5(\xi) = A_0[v_5(\xi)] + B_0 \)

From Case(2), inserting Eq.(44) into (34) and (40), admits to the new exact traveling wave solutions to Eqs.(29) as follows
\[
v(\xi) = a[-\frac{k^2 h_1}{4 \alpha_0^2 b_1} + \frac{1}{64 \alpha_0^2 b_1^2} \phi(\xi) + \frac{b_1}{\phi(\xi)} - \frac{\beta_0}{2\alpha}],
\]
\( u(\xi) = \alpha[a[-\frac{k^2 h_1}{4 \alpha_0^2 b_1} + \frac{1}{64 \alpha_0^2 b_1^2} \phi(\xi) + \frac{b_1}{\phi(\xi)} - \frac{\beta_0}{2\alpha}]]^2 + \beta_0[a[-\frac{k^2 h_1}{4 \alpha_0^2 b_1} + \frac{1}{64 \alpha_0^2 b_1^2} \phi(\xi) + \frac{b_1}{\phi(\xi)} - \frac{\beta_0}{2\alpha}]] + \gamma
\]
\( w(\xi) = A_0[a[-\frac{k^2 h_1}{4 \alpha_0^2 b_1} + \frac{1}{64 \alpha_0^2 b_1^2} \phi(\xi) + \frac{b_1}{\phi(\xi)} - \frac{\beta_0}{2\alpha}]] + B_0
\)

With the aid of Case (2) with Eqs.(54), admits as series of traveling wave solutions of Eqs.(29) as follows: If \( h_0 = h_1 = h_3 = 0, h_4 \neq 0 \), admits to exponential solution
\[
v_6(\xi) = a[-\frac{k^2 h_1}{4 \alpha_0^2 b_1} + \frac{1}{64 \alpha_0^2 b_1^2} \phi(\xi) + \frac{b_1}{\phi(\xi)} - \frac{\beta_0}{2\alpha}],
\]
\( u_6(\xi) = \alpha[v_6(\xi)]^2 + \beta_0[v_6(\xi)] + \gamma, \quad w_6(\xi) = A_0[v_6(\xi)] + B_0 \)

If \( h_1 = h_3 = h_4 = 0, h_0 > 0, h_2 > 0 \), admits to a triangular solution
\[
v_7(\xi) = a[-\frac{k^2 h_1}{4 \alpha_0^2 b_1} + \frac{1}{64 \alpha_0^2 b_1^2} \sqrt{-\frac{h_0}{h_2}} \sin(\sqrt{-h_2} \xi) + \frac{b_1}{\sqrt{-h_2}} \sqrt{-\frac{h_0}{h_2}} \sin(\sqrt{-h_2} \xi)] = \frac{\beta_0}{2\alpha},
\]
\( u_7(\xi) = \alpha[v_7(\xi)]^2 + \beta_0[v_7(\xi)] + \gamma, \quad w_7(\xi) = A_0[v_7(\xi)] + B_0 \)

If \( h_0 = h_1 = h_3 = h_4 > 0, h_2 > 0 \), admits to rational solution
\[
v_8(\xi) = a[-\frac{k^2 h_1}{4 \alpha_0^2 b_1} - \frac{1}{64 \alpha_0^2 b_1^2} \frac{1}{\sqrt{h_2}} \xi + \frac{b_1}{\sqrt{h_2}}] = \frac{\beta_0}{2\alpha}, \quad u_8(\xi) = \alpha[v_8(\xi)]^2 + \beta_0[v_8(\xi)] + \gamma
\]
\( w_8(\xi) = A_0[v_8(\xi)] + B_0 \)

If \( h_2 = h_4 = 0, h_3 > 0 \), we have admits to Weierstrass elliptic solution
\[
v_9(\xi) = a[-\frac{k^2 h_1}{4 \alpha_0^2 b_1} + \frac{1}{64 \alpha_0^2 b_1^2} \psi(\frac{\sqrt{h_3}}{2}, g_2, g_3) + \frac{b_1}{\psi(\frac{\sqrt{h_3}}{2}, g_2, g_3)}] = \frac{\beta_0}{2\alpha},
\]
\( u_9(\xi) = \alpha[v_9(\xi)]^2 + \beta_0[v_9(\xi)] + \gamma, \quad w_9(\xi) = A_0[v_9(\xi)] + B_0 \)

\( \text{I} \text{N} \text{S \ email \ for \ contribution: editor@nonlinearscience.org.uk} \)
When \( h_0 = h_4 = h_1 = 0, h_2 > 0 \), we have to a hyperbolic solution

\[
v_{10}(\xi) = a\left[-\frac{k^2 h_1}{4a b_1 a^3} + \frac{b_1}{\sqrt{h_2}}\right] - \frac{\beta_0}{2\alpha}, \
\]

\[
u_{10}(\xi) = a\left[v_{10}(\xi)\right]^2 + \beta_0[v_{10}(\xi)] + \gamma, \quad w_{10}(\xi) = A_0[v_{10}(\xi)] + B_0
\]

(59)

Proceeding as before, inserting Eq.(45) into (34) and (40), we have to the new exact travelling wave solutions for Eqs.(29) as follows

\[
v(\xi) = a\left[-\frac{k^2 h_1}{4a b_1 a^3} + \frac{b_1}{\phi(\xi)}\right] - \frac{\beta_0}{2\alpha}, \quad w(\xi) = A_0[a\left[-\frac{k^2 h_1}{4a b_1 a^3} + \frac{b_1}{\phi(\xi)}\right] - \frac{\beta_0}{2\alpha}] + B_0, \quad u(\xi) = \alpha[a\left[-\frac{k^2 h_1}{4a b_1 a^3} + \frac{b_1}{\phi(\xi)}\right] - \frac{\beta_0}{2\alpha}] + \gamma,
\]

(60)

With the aid of Case (3) with Eqs.(60), admits as series of travelling wave solutions of Eqs.(29) as follows: If \( h_1 = h_3 = h_4 = 0, h_0 > 0, h_2 > 0 \), we have to a hyperbolic solution

\[
v_{11}(\xi) = a\left[-\frac{k^2 h_1}{4a b_1 a^3} + \frac{b_1}{\sqrt{h_2}}\right] - \frac{\beta_0}{2\alpha}, \quad u_{11}(\xi) = a[v_{11}(\xi)]^2 - \beta_0[v_{11}(\xi)] + \gamma, \
\]

\[
w_{11}(\xi) = A_0[v_{11}(\xi)] + B_0
\]

(61)

If \( h_0 = h_1 = h_4 = 0, h_2 < 0 \), admits to a triangular solution

\[
v(\xi) = a\left[-\frac{k^2 h_1}{4a b_1 a^3} - \frac{b_1}{\sqrt{h_2}}\right] - \frac{\beta_0}{2\alpha}, \quad u_{12}(\xi) = a[v_{12}(\xi)]^2 - \beta_0[v_{12}(\xi)] + \gamma, \
\]

\[
w_{12}(\xi) = A_0[v_{12}(\xi)] + B_0, \quad \xi = k(x - ct)
\]

(62)

### 3.2 Example(2). The coupled Maccaris equations

A second instructive model is the coupled Maccaris equations [21]

\[
iQ_t + Q_{xx} + QR = 0, \quad R_t + R_y + |Q|^2 x = 0.
\]

(63)

In order to seek exact solutions of Eqs.(63), we suppose

\[
Q(x, y, t) = u(x, y, t)exp[i(kx + \alpha y + \lambda t + l)],
\]

(64)

where \( k, \alpha \) and \( \lambda \) are constants to be later determined, \( l \) is an arbitrary constant. Substituting Eq.(64) into Eqs.(63) and yields the following PDEs

\[
i(u_t + 2ku_x) + u_{xx} - (\lambda + k^2)u + uR = 0,
\]

(65)

\[
R_t + R_y + (u^2)_x = 0
\]

(66)

Using the transformation, we have

\[
u = u(\xi), R = R(\xi), \xi = w(x + \beta y - 2kt + xo),
\]

(67)

where \( \beta \) and \( w \) are constants to be determined later, \( xo \) is an arbitrary constant, Eqs.(65) and (66) become the following ODEs

\[
u^2 u - (\lambda + k^2)u + uR = 0,
\]

(68)

\[
(\beta - 2k)R_t + (u^2)_t = 0,
\]

(69)

where prime denotes the differential with respect to \( \xi \). Integrating Eq.(69) with respect to \( \xi \) and taking the integration constant as zero yields

\[
R = -\frac{1}{\beta - 2k}u^2(\xi)
\]

(70)

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Substituting Eq.(70) into (68), yields

$$\beta^2 u''(\xi) - (\lambda + k^2)u(\xi) - \frac{1}{\beta - 2k}u^3(\xi) = 0, \quad (71)$$

According to the proposed method, balancing the term $u^3(\xi)$ with term $u''(\xi)$ in Eq.(71) leads to $n = 1$. Therefore, we have

$$u(\xi) = a_0 + a_1\phi(\xi) + b_1\phi^{-1}(\xi) + c_1\phi'(\xi) + d_1\phi^{-1}(\xi)\phi'(\xi), \quad (72)$$

where $a_0, a_1, b_1, c_1, d_1, e$ and $k$ are constants to be determined. Substituting (72) along with (8) into (71) and collecting coefficients of $\phi^i$ and $\phi^{i'} \sum_{j=0}^{i} h_j \phi^j$, then setting each coefficient to zero to derive a set of algebraic equations for $a_0, a_1, b_1, c_1$ and $d_1$. Solving the system of algebraic equations with the aid of MAPLE, we obtain three different cases of the coefficients as follows

Case(1):

$$a_0 = \frac{h_1 \beta^2(\beta - 2k)}{2b_1}, h_2 = 0, d_1 = 0, h_3 = \frac{h_1 \beta^2(\beta - 2k)^2 + 4k\beta + 4k^2}{12b_1^2}, \quad (73)$$

$$k = k_1, b_1 = b_1, c_1 = 0, h_0 = 0, a_1 = 0, \lambda = -k^2, h_1 = h_1, h_4 = h_4,$$

Case(2):

$$c_1 = c_1, a_1 = a_1, h_1 = 0, h_4 = h_4, k = \frac{3\beta^3 h_4 - 4a_1^2}{6\beta^2 h_4}, b_1 = 0, h_2 = \frac{5a_1^2}{c_1^2}, d_1 = 0, h_3 = 0, a_0 = 0, \quad (74)$$

Case(3):

$$c_1 = 0, a_1 = a_1, h_1 = 0, h_0 = 0, k = \frac{\beta^3 - 2d_1^2}{2\beta^2}, h_4 = \frac{a_1^2}{d_1^2}, b_1 = 0, h_2 = 0, d_1 = d_1, \quad (75)$$

$$h_3 = 0, a_0 = 0, \lambda = \frac{\beta^6 - 4d_1^2\beta^3 + 4d_1^4}{4\beta^4}$$

With the aid of Case(1), inserting Eq.(73) into (70) and (64), we have a new exact traveling wave solutions for Eqs.(63) as follows

$$u(\xi) = \left[\frac{h_1 \beta^2(\beta - 2k)}{2b_1} \frac{b_1}{\phi(\xi)}\right], \quad (76)$$

$$Q(\xi) = \left[\frac{h_1 \beta^2(\beta - 2k)}{2b_1} + \frac{b_1}{\phi(\xi)}\right]e^{i(kx + \alpha y + \lambda t + l)}, \quad (77)$$

With the aid of Case (1) with Eqs.(70) and (64), admits as a series of travelling wave solutions of Eqs.(63) as follows: If $h_0 = h_2 = h_1 = 0, h_4 < 0$, admits to an exponential solution

$$u_1(\xi) = \left[\frac{h_1 \beta^2(\beta - 2k)}{2b_1} + \frac{b_1}{h_1 \beta^2(\beta - 2k) + 4k\beta + 4k^2}\right]e^{i(kx + \alpha y + \lambda t + l)}, \quad (78)$$

$$Q_1(\xi) = \left[\frac{h_1 \beta^2(\beta - 2k)}{2b_1} + \frac{b_1}{h_1 \beta^2(\beta - 2k) + 4k\beta + 4k^2}\right]e^{i(kx + \alpha y + \lambda t + l)}, \quad (79)$$

If $h_0 = h_1 = h_2 = 0$, we have a rational solution

$$u_2(\xi) = \left[\frac{h_1 \beta^2(\beta - 2k)}{2b_1} + \frac{b_1}{h_1 \beta^2(\beta - 2k)}\right], \quad (80)$$

$$Q_2(\xi) = \left[\frac{h_1 \beta^2(\beta - 2k)}{2b_1} + \frac{b_1}{h_1 \beta^2(\beta - 2k)}\right]e^{i(kx + \alpha y + \lambda t + l)}, \quad (81)$$

By using Case(2), inserting Eq.(74) into (70) and (64), we have a new exact traveling wave solutions for Eqs.(63) as follows

$$u(\xi) = \left[a_1 \phi(\xi) + c_1 \phi'(\xi)\right], \quad (82)$$

$$Q(\xi) = \left[a_1 \phi(\xi) + c_1 \phi'(\xi)\right]e^{i(kx + \alpha y + \lambda t + l)}, \quad (83)$$

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From Case (2), Eqs. (80) and (81), yields a series of travelling wave solutions of Eqs. (63) as: If \(h_1 = h_3 = h_4 = 0, h_0 > 0, h_2 > 0\), admits to a hyperbolic solution

\[
\begin{align*}
R_3(\xi) &= -\frac{1}{\beta - 2k} [u_3(\xi) + c_1 \phi' (\xi)]^2 \\
Q_3(\xi) &= [a_1 \sqrt{\frac{h_0}{h_2}} \sinh(\sqrt{\frac{h_2}{h_0}}) \alpha x + \beta t + l] \sinh(\sqrt{\frac{h_2}{h_0}}) \phi(\xi), \\
u_3(\xi) &= [a_1 \sqrt{\frac{h_0}{h_2}} \sinh(\sqrt{\frac{h_2}{h_0}}) \alpha x + \beta t + l] \phi(\xi).
\end{align*}
\]

If \(h_1 = h_3 = h_4 = 0, h_0 > 0, h_2 < 0\), admits to a triangular solution

\[
\begin{align*}
R_4(\xi) &= -\frac{1}{\beta - 2k} [u_4(\xi)]^2 \\
Q_4(\xi) &= [a_1 \sqrt{\frac{h_0}{h_2}} \sin(\sqrt{\frac{h_2}{h_0}}) \alpha x + \beta t + l] \sin(\sqrt{\frac{h_2}{h_0}}) \phi(\xi), \\
u_4(\xi) &= [a_1 \sqrt{\frac{h_0}{h_2}} \sin(\sqrt{\frac{h_2}{h_0}}) \alpha x + \beta t + l] \phi(\xi).
\end{align*}
\]

By using Case (3), with Eqs. (70) and (74), we have a new exact traveling wave solutions for Eqs. (63) as

\[
\begin{align*}
\phi(\xi) &= [a_1 \phi(\xi) + \frac{1}{\sqrt{h_2}} \sinh(\sqrt{\frac{h_2}{h_0}}) \phi(\xi)] e^{i(\alpha x + \beta t + l)}, \\
\psi(\xi) &= [a_1 \psi(\xi) + \frac{1}{\sqrt{h_2}} \sin(\sqrt{\frac{h_2}{h_0}}) \psi(\xi)] e^{i(\alpha x + \beta t + l)}, \\
\phi(\xi) &= [a_1 \phi(\xi) + \frac{1}{\sqrt{h_2}} \cosh(\sqrt{\frac{h_2}{h_0}}) \phi(\xi)] e^{i(\alpha x + \beta t + l)}.
\end{align*}
\]

From Case (3), if \(h_0 = h_1 = h_2 = h_3 = 0, h_4 > 0\), admits to a rational solution

\[
\begin{align*}
R_5(\xi) &= -\frac{1}{\beta - 2k} [u_5(\xi)]^2, \\
Q_5(\xi) &= [a_1 \sqrt{\frac{h_0}{h_2}} \phi(\xi)] e^{i(\alpha x + \beta t + l)}, \\
u_5(\xi) &= [a_1 \sqrt{\frac{h_0}{h_2}} \phi(\xi)] e^{i(\alpha x + \beta t + l)}, \\
\phi(\xi) &= [a_1 \phi(\xi)] e^{i(\alpha x + \beta t + l)}.
\end{align*}
\]

3.3 Example (3). The generalized Zakharov equations

The generalized Zakharov equations for the complex envelope \(\psi(x, t)\) of the high-frequency wave and the real low-frequency field \(v(x, t)\) in the form [14]

\[
\begin{align*}
i\psi_t + \psi_{xx} - 2\lambda |\psi|^2\psi + 2\psi v &= 0, \\
v_{tt} - v_{xx} + (|\psi|^2)_{xx} &= 0,
\end{align*}
\]

where the cubic term in Eq. (90) describes the nonlinear-self interaction in the high frequency subsystem, such a term corresponds to a self-focusing effect in plasma physics. The coefficient \(\lambda\) is a real constant that can be a positive or negative number. Let us assume the traveling wave solution of Eqs. (90) and (91) in the form

\[
\psi(x, t) = e^{i\eta} u(\xi), \quad v(x, t) = e^{i\eta} v(\xi), \quad \eta = \alpha x + \beta t, \quad \xi = k(x - 2\alpha t),
\]

where \(u(\xi)\) and \(v(\xi)\) are real functions, the constants \(\alpha, \beta\) and \(k\) are to be determined. Substituting (92) into Eqs. (90) and (91), we have

\[
\begin{align*}
k^2 u'' + 2uv - (\alpha^2 + \beta) u - 2\lambda u^3 &= 0, \\
k^2 (4\alpha^2 - 1)v'' + k^2 (u^2)'' &= 0
\end{align*}
\]
In order to simplify ODEs (93) and (94), integrating Eq.(94) once and taking integration constant as zero, we integrate, which yields

\[ u(\xi) = \frac{u^2}{(1 - 4\alpha^2)} + C_0, \quad \text{if } \alpha^2 \neq \frac{1}{4}, \quad (95) \]

where \( C_0 \)-integration constant. Inserting Eq.(95) into (93), we have

\[ k^2 u'' + [2C - \alpha^2 - \beta] u + 2\left[ \frac{1}{1 - 4\alpha^2} - \lambda \right] u^3 = 0 \quad (96) \]

Considering the homogeneous balance in Eq.(96), yields \( n = 1 \). Thus the ansatz solutions of Eq.(96) can be expressed by

\[ u(\xi) = a_0 + a_1 \phi(\xi) + b_1 \phi^{-1}(\xi) + c_1 \phi'(\xi) + d_1 \phi^{-1}(\xi) \phi'(\xi) \quad (97) \]

Proceeding as before, substituting (97) and inserting (8) into (96) and using MAPLE, yields as set of algebraic equations for \( a_0, a_1, b_1, c_1, d_1, k \) and \( \lambda \). Solving the system of algebraic equations, we have

\[ h_0 = 0, h_1 = 0, h_2 = -\frac{(\alpha^2 - 2C_0 + \beta)^2}{3k^4}, \quad c_1 = 0, \]

\[ h_1 = \frac{4(32\alpha^4 + 32\beta\alpha^2 + 8\alpha^2 - 9\lambda\alpha^2 - 64\lambda C_0\alpha^2 + 8\beta - 16C_0 + 18\lambda C_0 - 9\lambda)}{k^2 \lambda}, \]

\[ d_1 = \sqrt{-\frac{1 - 4\alpha^2}{4 - 4\lambda + 16\lambda \alpha^2}}, a_0 = \frac{-(\alpha^2 + 3\beta\alpha^2 + 2C_0 - 8C_0\alpha^2 + 4\alpha^2 - \beta)}{12(1 - \lambda + 4\lambda \alpha^2) \sqrt{4 - 4\lambda + 16\lambda \alpha^2}}, \]

\[ h_3 = \frac{(\beta^2 + 2\beta\alpha^2 - 4\lambda C_0\alpha^2 + 4C_0^2)\lambda}{108(1 - \lambda + 4\lambda \alpha^2)k^6} \]

Inserting Eq.(98) into Eqs.(92) and (95), admits to a new exact traveling wave solution as follows

\[ u(\xi) = \frac{-\alpha^2 + 4\beta\alpha^2 + 2C_0 - 8C_0\alpha^2 + 4\alpha^2 - \beta}{12(1 - \lambda + 4\lambda \alpha^2) \sqrt{4 - 4\lambda + 16\lambda \alpha^2}} \phi(\xi) \]

\[ \psi(\xi) = u(x)e^{i(\alpha x + \beta t)}, \psi(\xi) = \frac{u^2(\xi)}{1 - 4\alpha^2} + C_0, \quad \xi = k(x - 2\alpha t) \]

From Eqs.(99) and (100), we obtain a series of a new exact traveling wave solutions as follows: If \( h_0 = h_3 = h_4 = 0, h_2 < 0 \), admits to a triangular solution

\[ u_1(\xi) = [a_0 + a_1 \left( \frac{-h_1}{2h_2} + h_1 \right)] \sinh(\sqrt{-h_2} \xi) + d_1 \left( \frac{h_1}{2h_2} \right) \sinh(\sqrt{-h_2} \xi)^{-1} \sqrt{-h_2} \frac{h_1}{2h_2} \cos(\sqrt{-h_2} \xi)], \]

\[ v_1(\xi) = u_1(\xi)e^{i(\alpha x + \beta t)}, \quad v_1(\xi) = \frac{u_1^2(\xi)}{1 - 4\alpha^2} + C_0 \]

If \( h_0 = h_3 = h_4 = 0, h_2 > 0 \), admits to a hyperbolic solution

\[ u_2(\xi) = [a_0 + a_1 \left( \frac{h_1}{2h_2} + h_1 \right)] \sinh(2\sqrt{h_2} \xi) + d_1 \left( \frac{h_1}{2h_2} \right) \sinh(2\sqrt{h_2} \xi)^{-1} \sqrt{h_2} \frac{h_1}{2h_2} \cosh(2\sqrt{h_2} \xi)], \]

\[ v_2(\xi) = u_2(\xi)e^{i(\alpha x + \beta t)}, \quad v_2(\xi) = \frac{u_2^2(\xi)}{1 - 4\alpha^2} + C_0 \]
If $h_0 = h_1 = h_4 = 0, h_2 > 0$, admits to a hyperbolic solution

$$u_3(\xi) = a_0 - a_1 \frac{h_2}{h_3} \text{sech}^2\left(\frac{\sqrt{h_2}}{2} \xi\right) + d_1 \left[\frac{h_2}{h_3} \text{sech}^2\left(\frac{\sqrt{h_2}}{2} \xi\right)\right]^{-1} \frac{h_2^{3/2}}{h_3} \text{sech}^2\left(\frac{\sqrt{h_2}}{2} \xi\right) \tanh\left(\frac{\sqrt{h_2}}{2} \xi\right),$$

$$\psi_3(\xi) = u_3(\xi) e^{i(\alpha x + \beta t)}, \quad v_3(\xi) = \frac{u_3^2(\xi)}{1 - 4\alpha^2} + C_0 \quad (104)$$

If $h_0 = h_1 = h_2 = 0, h_4 < 0$, admits to an exponential solution

$$u_4(\xi) = a_0 + a_1 \frac{h_3}{2h_4} \exp\left(\frac{h_3}{2\sqrt{-h_4}} \xi\right) + d_1 \left[\frac{h_3}{2h_4} \exp\left(\frac{h_3}{2\sqrt{-h_4}} \xi\right)\right]^{-1} \frac{-h_2^2}{4h_4\sqrt{-h_4}} \exp\left(\frac{h_3}{2\sqrt{-h_4}} \xi\right)$$

$$\psi_4(\xi) = u_4(\xi) e^{i(\alpha x + \beta t)}, \quad v_4(\xi) = \frac{u_4^2(\xi)}{1 - 4\alpha^2} + C_0 \quad (105)$$

If $h_0 = h_1 = h_2 = 0, h_4 < 0, h_2 < 0$, admits to a triangular solution

$$u_5(\xi) = a_0 + a_1 \sqrt{-\frac{h_2}{h_4}} \text{sech}\left(\sqrt{-h_2} \xi\right) + d_1 \left[\sqrt{-\frac{h_2}{h_4}} \text{sech}\left(\sqrt{-h_2} \xi\right)\right]^{-1} \sqrt{-h_2} \sqrt{-\frac{h_2}{h_4}} \text{sech}\left(\sqrt{-h_2} \xi\right) \tan\left(\sqrt{-h_2} \xi\right),$$

$$\psi_5(\xi) = u_5(\xi) e^{i(\alpha x + \beta t)}, \quad v_5(\xi) = \frac{u_5^2(\xi)}{1 - 4\alpha^2} + C_0 \quad (106)$$

### 4 Conclusions

An extended algebraic method with a computerized symbolic computation has been proposed to obtain new exact solutions to three nonlinear evolution equations arising in nonlinear mathematical physics. The validity of this method has been tested by applying it successfully to generalized Hirota-Satsuma coupled KdV system, coupled Maccari equations and generalized-Zakharov equations. As a result, the exact traveling wave solution is obtained include soliton solutions, rational solutions, triangular periodic solutions, Jacobi and Weierstrass doubly periodic wave solutions. Among them, the Jacobi elliptic periodic wave solutions exactly degenerate to the soliton solutions at a certain limit condition. It is worthwhile to mention that the method is straightforward and concise, and it can also be applied to other nonlinear evolution equations in physics. This is our task in future.

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### References


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