The Application of Contraction Theory in Synchronization of Coupled Chen Systems

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Abstract: In this brief, the contraction theory is introduced, and applied to the synchronization of coupled nonlinear systems where a contracting virtual system of the coupled systems is needed. Then based on contraction theory, synchronization of coupled Chen system is invested. Finally, Simulink's numeration shows that completely synchronization, co-existence synchronization and anti-synchronization of coupled Chen systems is well achieved.

Keywords: contraction theory, invariant time-varying set, chaos synchronization

1 Introduction

Since the revolutionary work of Pecora and Carroll[1], synchronization phenomenon has formed a new body of research activities which is at the fore front of recent application topics in nonlinear systems[2-5], and it has fascinated many scientists from different fields, various schemes of synchrony were obtained.

A typical problem is to guarantee the synchronization of a coupled of nonlinear systems, so that all solutions asymptotically converge toward the same common evolution. We know that synchronization problems have to do with stability problems. Rigorous stability results were obtained by constructing appropriate Lyapunov functions such as in completely synchronization (CS)[1], phase synchronization(PS) [6], anti-synchronization(AS) [7], projective synchronization (PjS) [8,9], et al. Asymptotic stability of all solutions of a nonlinear dynamical system are due to the Russian mathematician Demidovich[10]. Several decades after Demidovich publications, the interest on the stability properties of trajectories with respect to each other is revived: the incremental stability [11] and contraction theory related to these concepts. The contraction theory has been established as an effective tool for analyzing the convergence behavior of nonlinear systems in state space form. It was successfully applied to both nonlinear control and observer problems [12, 13].

Motivated by contraction theory, we conceive that if a virtual system of coupled nonlinear systems is contracting, then the coupled systems would be synchronized. We invest Chen system [14] based on contraction theory, then numeric result demonstrate completely synchronization, also $x$-variable $y$-variable anti-synchronization and $z$-variable synchronization are achieved.

2 Problem description

In this section, contraction Theory is introduced first, and then its application in synchronization of coupled nonlinear system is achieved on condition that there exists a virtual system which is contracting.

A nonlinear dynamical system is called contracting if initial conditions or temporary perturbations decay exponentially fast. We consider general deterministic systems of the form $\dot{x} = f(x)$, where $x \in \mathbb{R}^m$, $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$. All quantities are assumed to be real and sufficiently smooth. Recall that a virtual displacement is an infinitesimal displacement at fixed time. Formally, it defines a linear tangent differential form [15]. Considering two neighboring trajectories of the flow, the virtual displacement $\delta x$ between them and the virtual velocity $\delta \dot{x}$, we have

$$\delta \dot{x} = \frac{\partial f(x)}{\partial x} \delta x$$ (1)
From (1), we get
\[
\frac{d(\delta x^T \delta x)}{dt} = 2\delta x^T \frac{\partial f(x)}{\partial x} \delta x \leq 2\lambda_{\text{max}}\delta x^T \delta x
\]  
(2)

where \(\lambda_{\text{max}}\) is the largest eigenvalue of the symmetric part of the Jacobian \(\frac{\partial f(x)}{\partial x}\). Hence, if \(\lambda_{\text{max}}\) in (2) is uniformly strictly negative, any infinitesimal length exponentially converges to zero. By path integration, this immediately implies that the length of any finite path in phase space exponentially converges to zero, i.e., that distances shrink in phase space. The contraction principle, which is derived in [15], can be stated as follows:

**Theorem 1:** Let \(x(t)\) and \(\tilde{x}(t)\) be two generic trajectories of (1). Let \(M_t : B_\varepsilon(x(t))\) and let \(C \subseteq \mathbb{R}^n\) be a contracting region in phase space, which is defined as:

\[
C := \left\{ x \in \mathbb{R}^n : \frac{1}{2} \left( \frac{\partial f}{\partial x} + \frac{\partial f^T}{\partial x} \right) \leq -\beta I, \beta > 0, \forall t \in \mathbb{R}^+ \right\}
\]

If \(\tilde{x}(t)\) satisfies 1) \(\tilde{x}(t_0) \in M_0\), and 2) \(M_t \subseteq C, \forall t \in \mathbb{R}^+\), then a) \(\tilde{x}(t) \in M_t, \forall t \in \mathbb{R}^+\), and b) \(\delta x^T \delta x \leq k\delta x_0^T \delta x_0 e^{-\beta t}, k \geq 1, \beta > 0, \forall t \in \mathbb{R}^+, \) and vice versa.

Thus, all the solutions of (1) starting from any initial condition inside the contraction region will remain in \(C\) and exponentially converge to a single trajectory. Take note of the following: 1) In Theorem 1, \(M_t\) is an invariant time-varying set; and 2) the dynamics along a chaotic attractor are not contracting as they are characterized by local exponential divergence of nearby trajectories.

To apply the contraction theory to synchronization of nonlinear systems, the construction of a virtual system is needed [13]. Such a system depends on the state variables of the systems and on some virtual state variables. The proof of the contracting property with respect to these virtual state variables immediately implies synchronization.

From theorem 1 above, we can easily obtain a corollary below:

**Corollary:** Suppose two \(n\)-dimension nonlinear systems:

\[
\begin{align*}
\dot{Y} &= F(Y) + H(Z) - H(Y) \\
\dot{Z} &= F(Z) + H(Y) - H(Z)
\end{align*}
\]

where \(Y = (y_1, y_2, \ldots, y_n), Z = (\alpha_1 y_1, \alpha_2 y_2, \ldots, \alpha_n y_n) \in \mathbb{R}^n, \) and \(\alpha_i \in R(i = 12, \ldots, n), F : \mathbb{R}^n \rightarrow \mathbb{R}^n, H\) is some output function. Then a virtual system of them can be constructed:

\[
\phi(X, Y, Z) = F(X) - 2H(X) + H(Y) + H(Z)
\]

So, \(\dot{Y} = \phi(Y, Y, Z), \dot{Z} = \phi(Z, Y, Z).\) If the virtual system is contracting according to virtual \(X\)-variable, then the coupled systems are synchronized.

### 3 The investigation of Chen system’s synchronization based on contraction theory

All we are known that the noticeable Chen system [15] depicts as:

\[
\begin{align*}
\dot{x}_1 &= a(y_1 - x_1) \\
\dot{y}_1 &= (c - a)x_1 - x_1z_1 + cy_1 \\
\dot{z}_1 &= x_1y_1 - bz_1
\end{align*}
\]  
(3)

and it’s another identical system is:

\[
\begin{align*}
\dot{x}_2 &= a(y_2 - x_2) \\
\dot{y}_2 &= (c - a)x_2 - x_2z_2 + cy_2 \\
\dot{z}_2 &= x_2y_2 - bz_2
\end{align*}
\]  
(4)

where \((x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3, a, b, c\) are the system’s parameters. Let \((x', y', z') = (\alpha_1 x_2, \alpha_2 y_2, \alpha_3 z_2)\) then system (4) is...
\[
\begin{align*}
\dot{x}' &= a(x_{1}' - x') \\
\dot{y}' &= (c - a)x_{1}' + cy' + cy \quad (\text{with } a, c > 0) \\
\dot{z}' &= \frac{x_{1}z_{2} + cy_{1}}{\alpha_{1}\alpha_{2}}x'_{1}' - bz' \\
\end{align*}
\] (5)

Based on the contraction theory and the corollary above, if we can construct a virtual system of system (3) and (5), furthermore, the virtual system is contracting, then the trajectories of (3) and (5) will converge to a single trajectory. Namely, system (3) and (5) will be synchronized.

3.1 The completely synchronization of Chen system

When \(\alpha_{1} = \alpha_{2} = \alpha_{3} = 1\), system (5) is system (4). Assume that there are couplings between system (3) and (4), which could be described as bellow:

\[
\begin{align*}
\dot{x}_{1} &= a(y_{1} - x_{1}) + \varepsilon_{x}(k_{1}(x_{2} - x_{1})) \\
\dot{y}_{1} &= (c - a)x_{1} - x_{2}z_{1} + cy_{1} + \varepsilon_{y}(k_{2}(y_{2} - y_{1})) \\
\dot{z}_{1} &= x_{1}y_{1} - b_{2}z_{1} + \varepsilon_{z}(k_{2}z_{2} - z_{1}) \\
\end{align*}
\] (6)

\[
\begin{align*}
\dot{x}_{2} &= a(y_{2} - x_{2}) + \varepsilon_{x}(k^{\ast}_{1}(x_{1} - x_{2})) \\
\dot{y}_{2} &= (c - a)x_{2} - x_{2}z_{2} + cy_{2} + \varepsilon_{y}(k^{\ast}_{2}(y_{2} - y_{1})) \\
\dot{z}_{2} &= x_{2}y_{2} - b_{2}z_{2} + \varepsilon_{z}(k^{\ast}_{2}z_{2} - z_{1}) \\
\end{align*}
\] (7)

where \(\varepsilon_{x}=1\) denotes the coupling that is active on the \(x\)-variable, otherwise \(\varepsilon_{x}=0\); \(k_{i}\) and \(k^{\ast}_{i}\) are the coupling strength respectively.

Then a virtual system can be constructed as follows:

\[
\begin{align*}
\dot{x} &= a(y - x) + \varepsilon_{x}(k_{1}x_{2} + k^{\ast}_{1}x_{1} - (k_{1} + k^{\ast}_{1})x) \\
\dot{y} &= (c - a)x - xz + cy + \varepsilon_{y}(k_{2}y_{2} + k^{\ast}_{2}y_{1} - (k_{2} + k^{\ast}_{2})y) \\
\dot{z} &= xy - bz + \varepsilon_{z}(k_{3}z_{2} + k^{\ast}_{3}z_{1} - (k_{3} + k^{\ast}_{3})z) \\
\end{align*}
\] (8)

Let system (8) be: \(\dot{X} = \Psi(X)\), where \(X = (x, y, z) \in R^{3}\).

**Theorem 2:** Such that the conditions are satisfied: \(a = a_{1} + k_{1} > 0, c = c_{1} + k_{2} > 0, b = b_{1} + k_{3} > 0\). Where \(\tilde{k}_{1} = (k_{1} + k^{\ast}_{1})\varepsilon_{x}, \tilde{k}_{2} = (k_{2} + k^{\ast}_{2})\varepsilon_{y}, \tilde{k}_{3} = (k_{3} + k^{\ast}_{3})\varepsilon_{z}\), then system (8) is contracting, and its contracting region is:

\[
C = \left\{ (x, y, z) : \frac{y^{2}}{4a^{*}c^{*}} + \frac{(z - c)^{2}}{4a^{*}c^{*}} < 1, |z - c| < 2\sqrt{a^{*}c^{*}}, x \in R \right\}
\]

**Proof:** considering the velocity of solution trajectory in system (8): \(\dot{X} = \frac{\partial \Psi}{\partial X} \delta X\)

Let \(A = \frac{\partial \Psi}{\partial X} = \begin{pmatrix}
-a - (k_{1} + k^{\ast}_{1})\varepsilon_{x} & a & 0 \\
-c - (k_{2} + k^{\ast}_{2})\varepsilon_{y} & c & -x \\
-b - (k_{3} + k^{\ast}_{3})\varepsilon_{z} & 0 & b
\end{pmatrix}\), and \(-\Lambda = -\frac{1}{2}(A^{T} + A)\)

We know that \(-\Lambda\) is a real symmetric matrix, and its eigenvalues are all real, for \(a^{*} > 0, b^{*} > 0, c^{*} > 0\), and if \(\frac{y^{2}}{4a^{*}c^{*}} + \frac{(z - c)^{2}}{4a^{*}c^{*}} < 1\) holds, then the following conditions holds:

\[
\begin{align*}
a^{*} > 0 \\
a^{*}e^{2} - \frac{(z - c)^{2}}{4} > 0 \\
a^{*}b^{2}c^{2} - e^{2}b^{2}/4 - b^{2} > 0
\end{align*}
\]

\(-\Lambda\) is positive defined, at the same time \(\Lambda\) is negative defined. When \(c > 0\), it is noticeable that \(\varepsilon_{y} = 1\), which means that the coupling between \(y\)-variables must exist, or else, the largest eigenvalue of the matrix \(\Lambda\) is positive. So,

\[
\frac{d(\delta X^{T} \delta X)}{dt} = 2\delta X^{T} \frac{\partial \Psi(X)}{\partial X} \delta X \leq 2\lambda_{max} \delta X^{T} \delta X < 0.
\]

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$- \lambda_{\text{max}}$ is the largest eigenvalue of the symmetric part of the Jacobin $\frac{\partial \Psi(X)}{\partial X}$, according to theorem 1, system (8) is contracting in region

$$C : \left\{ (x, y, z) : \frac{y^2}{4a^*b^*} + \frac{(z-c)^2}{4a^*c^*} < 1, |z-c| < 2\sqrt{a^*c^*}, x \in R \right\}$$

From the proof, we know that the choosing of coupling strength $\tilde{k}_1, \tilde{k}_2, \tilde{k}_3$ is independent on each other, and if $\tilde{k}_1$ is defined, then $\tilde{k}_2$ and $\tilde{k}_3$ define the range of the space state $z$-variable and $y$-variable respectively. Especially, when $\tilde{k}_1 \to \infty$, contracting region $C = R^3$; when $\tilde{k}_3 \to \infty$, contracting region

$$C : \left\{ (x, y, z) : |z-c| < 2\sqrt{a^*c^*}, x \in R, y \in R \right\}.$$

Since system (8) is contracting, which implies that the synchronization of system (3) and (5) is achieved, resultantly, completely synchronization is achieved in system (3) and (4).

3.2 The co-existence synchronization and anti-synchronization in coupled Chen system

When $\alpha_1 = \alpha_2 = -1, \alpha_3 = 1$, however, system (5) transforms into the form of system (4). The virtual system can be chosen as above, and it is also contracting in the same region. So the two systems (3) and (5) are synchronized, namely, in system (3) and (4), only the $z$-variable synchronizes and the remained variables anti-synchronize.

4 Simulink numeration

To validate contraction theory apply in synchronizing nonlinear system, we choose $a = 35, b = 3, c = 28$, where Chen system exhibits chaotic behavior. Supposed that only the $y$-coupling in the coupled Chen systems exist, namely, $\varepsilon_y = 1, \varepsilon_x = 0$ and $\varepsilon_z = 0$. 

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(i) When $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $k_2 = -7$, $k_2^* = 49$, and initial conditions $(x_{10}, y_{10}, z_{10}) = (30, 16, 15)$, $(x_{20}, y_{20}, z_{20}) = (-20, -15, 35) \in \left\{ (x, y, z) : \frac{y^2}{420} + \frac{(z-28)^2}{1960} < 1, |z - 28| < 2\sqrt{490}, x \in \mathbb{R} \right\}$.

For completely synchronization is greatly and quickly achieved less than 1 second, by Simulink we only give the figures (Fig1.a~Fig1.c) in 2 seconds as follows.

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(ii) When $\alpha_1 = \alpha_2 = -1$, $\alpha_3 = 1$, $k_2 = -7$ and $k_2^* = 49$, initial conditions $(x_{10}, y_{10}, z_{10}) = (15, 16, 30)$, $(x_{20}, y_{20}, z_{20}) = (-20, -15, 35) \in \left\{ (x, y, z) : \frac{y^2}{420} + \frac{(z-28)^2}{1960} < 1, |z - 28| < 2\sqrt{490}, x \in \mathbb{R} \right\}$. Then $x$-variable and $y$-variable anti-synchronize, but $z$-variable synchronizes, using Simulink the figures (Fig2.a~Fig2.c) are depicted in 4 seconds as follows.

5 Conclusion

In this brief, contraction theory is introduced; a virtual system of coupled nonlinear systems is conceived, then under some conditions the virtual system of Chen systems is proved contracting, at the same time, the coupled systems are synchronized. At last, from Simulink’s numeration, the coupled Chen system’s synchronization and co-existence synchronization and anti-synchronization are well achieved.

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