Hausdorff Dimension of Generalized Sierpinski Carpet

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Abstract. From a convex quadrangle we construct a generalized Sierpinski carpet not self-similar, and prove that its Hausdorff dimension is $\frac{\log 8}{\log 3}$ by using a bi-Lipschitz mapping.

Keywords: Sierpinski carpet; Hausdorff dimension; bi-Lipschitz mapping

1 Introduction

Recently the research of fractal geometry ([1]-[4]) and chaotic phenomena([5]-[7]) are very interesting. We recall the standard result on self-similar set ([1],[4]). A mapping $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a contractive similitude with ratio $\rho \in (0, 1)$, if $|S(x) - S(y)| = \rho|x - y|$ for all $x, y \in \mathbb{R}^n$. Suppose $\{S_i\}_{i=1}^m$ is a family of similitudes with contraction ratios $\rho_1, \cdots, \rho_m$. Then there exists a unique compact set $E$ such that $E = \bigcup_{i=1}^m S_i(E)$, here $E$ is called the self-similar set generated by $\{S_i\}_{i=1}^m$. We say that the open set condition (OSC) holds for $\{S_i\}_{i=1}^m$, if there exists a bounded non-empty open set $U \subset \mathbb{R}^n$ such that $\bigcup_{i=1}^m S_i(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for $i \neq j$. Let $\dim_H E = s$, where $\dim_H E$ denotes the Hausdorff dimension of $E$. If OSC holds, then $\rho_1^s + \rho_2^s + \cdots + \rho_m^s = 1$.

We also recall the Sierpinski carpet ([2], [4]), a classical self-similar fractal set in the plane $\mathbb{R}^2$. Take a unit square $[0, 1]^2$, for eight points $(x_i, y_i)$ from the set $\{0, 1, 2\} \times \{0, 1, 2\} \setminus \{1, 1\}$, let $\phi_i(x, y) = [(x, y) + (x_i, y_i)]/3$ with ratio $1/3$. Then the Sierpinski carpet $F$ of $\mathbb{R}^2$ is the self-similar set generated by $\{\phi_i\}_{i=1}^8$, where OSC holds with $U = (0, 1)^2$ and thus $\dim_H F = \log 8/\log 3$.

![Figure 1: The steps of generating the Sierpinski carpet](image)

In this paper, we deal with the Hausdorff dimension of generalized Sierpinski carpet defined as follows.

Take a convex quadrangle $Q_0$, we trisect every side of the quadrangle and connect the corresponding trisection points of opposite sides, then we get a division of $Q_0$ into nine small quadrangles with their interiors pairwise disjoint, delete the small quadrangle which is right in the center of $Q_0$. By this “rule”, we then repeat the process for the eight remaining sub-quadrangles, and obtain eight sub-sub-quadrangles for each remaining sub-quadrangles, continue this process inductively.

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In this manner, we get a sequence of compact sets \( \{ K_l \}_{l=0}^{\infty} \), where \( K_0 = Q_0 \), and \( K_l \) is in general a union of \( 8^l \) convex quadrangles which are obtained from \( K_{l-1} \) by applying the basic “rule” mentioned above to each of the \( 8^{l-1} \) quadrangles of \( K_{l-1} \). Then \( K_l \subseteq K_{l-1} \) for all \( l \geq 1 \). Set

\[
K = \bigcap_{l=0}^{\infty} K_l,
\]

which is called the \textit{generalized Sierpinski carpet} of convex quadrangle \( Q_0 \).

The generalized Sierpinski carpet is not a self-similar set (see figure 2). How to calculate its Hausdorff dimension? In this paper, we will prove the following main result by using a bi-Lipschitz mapping (See [3], [8]-[11] for Lipschitz equivalence).

\textbf{Theorem 1.1} Suppose \( Q_0 \) is a convex quadrangle. Let \( K \) be the generalized Sierpinski carpet of \( Q_0 \). Then \( \dim_H K = \log 8 / \log 3 \).

\section{Preliminaries}

Suppose \((X, d)\) and \((Y, D)\) are metric spaces. A bijection \( f : (X, d) \to (Y, D) \) is said to be a bi-Lipschitz mapping, if there are constants \( C_1, C_2 > 0 \) such that for all \( x_1, x_2 \in X \),

\[
C_1 d(x_1, x_2) \leq D(f(x_1), f(x_2)) \leq C_2 d(x_1, x_2).
\]

The following lemma is also a standard result in fractal geometry (see for example [1], [2] and [4]).

\textbf{Lemma 2.1} If \( f : (X, d) \to (Y, D) \) is a bi-Lipschitz mapping, then \( \dim_H A = \dim_H f(A) \) for all \( A \subset X \).

Given points \( M, N \subset \mathbb{R}^2 \) and the line segment \( MN \), suppose a point \( L \) lies in \( MN \). If there exists \( \lambda \in [0, 1] \) such that \( |ML| : |MN| = \lambda \), then

\[
L = [(1 - \lambda)M + \lambda N] \subset \mathbb{R}^2,
\]

and the point \( L \) is called the point of definite proportion \( \lambda \) in \( MN \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{bi-Lipschitz_mapping.png}
\caption{bi-Lipschitz mapping from \([0,1]^2\) to convex hull of \( Q_0 \)}
\end{figure}

Take a unit square \([0,1]^2\) and a convex quadrangle \( Q_0 \) respectively (see Figure 3). Given any point \((\alpha, \beta) \in [0,1]^2\), as in Figure 3, in the quadrangle \( Q_0 \), we connect the points of definite proportion \( \alpha \) in \( AB \),...
and $DC$ to get a line segment, which is called $\alpha$ quasi-vertical segment; similarly, we connect the points of definite proportion $\beta$ in $AD$ and $BC$ to get a line segment, which is called $\beta$ quasi-horizontal segment. The point of intersection of $\alpha$ quasi-vertical segment and $\beta$ quasi-horizontal segment is denoted by $[\alpha, \beta]$.

Denote by $\text{cov}(Q_0)$ the convex hull of quadrangle $Q_0$.

Let $f : [0, 1]^2 \to \text{cov}(Q_0)$ be defined by

$$f(\alpha, \beta) = [\alpha, \beta]$$

for any $(\alpha, \beta) \in [0, 1]^2$. (3)

In order to prove Theorem 1, we need some conclusions in plane geometry.

![Figure 4: GH: $\alpha$ quasi-vertical segment; EF: $\beta$ quasi-horizontal segment](image)

**Lemma 2.2** Given a convex quadrangle, $[\alpha, \beta]$ is not only the point of definite proportion $\alpha$ in $\beta$ quasi-horizontal segment, but also the point of definite proportion $\beta$ in $\alpha$ quasi-vertical segment.

**Proof.** As in Figure 4, we have

$$E = (1 - \beta)A + \beta D, \quad F = (1 - \beta)B + \beta C,$$

$$H = (1 - \alpha)A + \alpha B, \quad G = (1 - \alpha)D + \alpha C.$$  

It suffices to verify that the point of definite proportion $\alpha$ in $EF$ is just the point of definite proportion $\beta$ in $HG$. In fact, we have

$$[(1 - \alpha)(1 - \beta)]A + [\alpha(1 - \beta)]B + (\alpha\beta)C + [(1 - \alpha)\beta]D$$

$$= (1 - \alpha)(1 - \beta)A + \alpha(1 - \beta)B + \beta C$$

$$= (1 - \beta)(1 - \alpha)A + \alpha B + \beta [(1 - \alpha)D + \alpha C]$$

that is $(1 - \alpha)E + \alpha F = (1 - \beta)H + \beta G$. □

Fix any vertex $x$ of $Q_0$, for any $(\alpha, \beta) \in [0, 1]^2$, let $\theta_x(\alpha, \beta)$ denote the included angle between the $\alpha$ quasi-vertical segment and $\beta$ quasi-horizontal segment such that the point $x$ is contained in the corresponding sector determined by the angle.

**Lemma 2.3** Given a convex quadrangle $Q_0$ with vertexes $\{A, B, C, D\}$, there exists $\theta_0 > 0$ such that $\theta_x(\alpha, \beta) \geq \theta_0$ for any $(\alpha, \beta)$ and any vertex $x$.

**Proof.** Note that $\theta_x(\alpha, \beta)$ is a continuous function with respect to $(\alpha, \beta) \in [0, 1]^2$. Here $[0, 1]^2$ is compact set, and $\theta_x(\alpha, \beta) > 0$ for each $(\alpha, \beta)$, and thus we can get the minimal value $\min_{(\alpha, \beta)\in[0,1]^2} \theta_x(\alpha, \beta) > 0$ of $\theta_x(\alpha, \beta)$. Let

$$\theta_0 = \min_{x \in \{A, B, C, D\}} \min_{(\alpha, \beta)\in[0,1]^2} \theta_x(\alpha, \beta) > 0.$$  

And thus $\theta_x(\alpha, \beta) \geq \theta_0$. □
Lemma 2.4 Fix $\theta^* > 0$. Suppose $a, b, c$ are side lengths of any triangle with the angle $\theta \geq \theta^*$ (see Figure 5). Then there exists a constant $q = q(\theta^* ) > 0$ such that

\[ q(a + b) \leq c \leq a + b. \]

Proof. By triangle inequality, we have

\[ c \leq a + b. \]

On the other hand, we shall distinguish two cases:

**Case 1:** $\theta^*$ is an acute angle.

In this case $\cos \theta^* > 0$. By cosine law, we have

\[
c = \sqrt{a^2 + b^2 - 2ab \cos \theta} \geq \sqrt{a^2 + b^2 - 2ab \cos \theta^*} 
\geq \sqrt{(1 - \cos \theta^*)(a^2 + b^2)} + (a - b)^2 \cos \theta^* 
\geq \sqrt{(1 - \cos \theta^*)/(a + b)} = \sin(\theta^*/2)(a + b).
\]

Take $q = q(\theta^*) = \sin(\theta^*/2)$, the conclusion holds.

**Case 2:** $\theta^*$ is not an acute angle.

In this case $\cos \theta \leq 0$, we have

\[
c = \sqrt{a^2 + b^2 - 2ab \cos \theta} \geq \sqrt{(a^2 + b^2)} \geq \sqrt{1/2}(a + b)
\]

Take $q = q(\theta^*) = \sqrt{1/2}$, the conclusion holds. \(\Box\)

3 Proof of Theorem 1

Let $F$ be the Sierpinski carpet and $K$ the generalized Sierpinski carpet. It follows from Lemma 2.2 that $f(F) = K$, where $f$ is defined in formula (3).

Since $\dim_H F = \log 8/\log 3$, by Lemma 2.1, it suffices to prove that $f : [0, 1]^2 \to \text{cov}(Q_0)$ is a bi-Lipschitz mapping, that means there exist constants $C_1, C_2 > 0$ such that for any $(\alpha, \beta), (\alpha', \beta') \in [0, 1]^2$,

\[ C_1 d((\alpha, \beta), (\alpha', \beta')) \leq d([\alpha, \beta], [\alpha', \beta']) \leq C_2 d((\alpha, \beta), (\alpha', \beta')) ,\]

where $d$ is the Euclidean metric in $\mathbb{R}^2$.

Since $d((\alpha, \beta), (\alpha', \beta')) = \sqrt{(\alpha - \alpha')^2 + (\beta - \beta')^2}$, we have

\[ |\alpha - \alpha'| + |\beta - \beta'| \leq d((\alpha, \beta), (\alpha', \beta')) \leq |\alpha - \alpha'| + |\beta - \beta'| ,\]

which implies we need only to find constants $\delta_1, \delta_2 > 0$ such that

\[ \delta_1 |\alpha - \alpha'| + |\beta - \beta'| \leq d([\alpha, \beta], [\alpha', \beta']) \leq \delta_2 (|\alpha - \alpha'| + |\beta - \beta'|) .\]

Figure 5: Estimation of $c$ in terms of $a + b$
Proof of formula (4):
Let $\varsigma(\alpha)$ denote the length of $\alpha$ quasi-vertical segment. Then $\varsigma(\alpha)$ is a continuous function with respect to $\alpha \in [0, 1]$, and thus there exist constants $v_1, v_2 > 0$ such that
\[
 v_1 \leq \varsigma(\alpha) \leq v_2.
\] (5)

Let $\tau(\beta)$ denote the length of $\beta$ quasi-horizontal segment. Then $\tau(\beta)$ is a continuous function with respect to $\beta \in [0, 1]$, and thus there exist constants $h_1, h_2 > 0$ such that
\[
 h_1 \leq \tau(\beta) \leq h_2.
\] (6)

As in Figure 6, by Lemma 2.2, we have
\[
a = |\alpha - \alpha'| \cdot \tau(\beta),
\]
\[
b = |\beta - \beta'| \cdot \varsigma(\alpha').
\] (7)

It follows from Lemma 2.3 and 2.4 that
\[
q(a + b) \leq d([\alpha, \beta], [\alpha', \beta']) = c \leq a + b,
\] (8)

where $q > 0$ is a constant.

Let $\delta_1 = q \min(v_1, h_1)$ and $\delta_2 = \max(v_2, h_2)$, then (4) follows from (5), (6), (7) and (8).

Therefore, Theorem 1 is proved.

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