Boundary Control of the Kuramoto-Sivashinsky Equation with an External Excitation

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Abstract: We study a dynamic system described by the Kuramoto-Sivashinsky equation with an external excitation \(f\) posed on a finite domain. Firstly it shows that under the given boundary feedback conditions it admits a unique solution and the solution is stable. Secondly it proves that if the external excitation \(f\) is a time periodic function, then the system under the boundary conditions admits a unique time periodic solution and its period is the same as the \(f'\)s.

Keywords: Kuramoto-Sivashinsky equation; external excitation; stability; boundary control; boundary feedback condition

1 Introduction

Boundary control is one kind of distributed parameter controls, which has been emphasized in the control theory and has been extensively studied and developed. Recently more and more people take notice on the boundary control of Burgers equation, KdV equation, KdVB equation, C-H equation [26, 27], and K-S equation [28]. There have been very rich researches in this field. For Burgers equation, recent references include J.A.Burns\([7]\), C.I. Byrnes\([8]\), H.Choi\([9]\). Byrnes et al \([8]\) got that a linear boundary controller achieves local exponential stability (the initial condition needs to be small in \(L^2\)) and later in \([26]\) the authors of \([8]\) extend their results to establish semi-global stabilization using the same linear boundary feedback. Van ly et al \([5]\) improved this result they extend it to \(L^\infty\) but remain local. Miroslav Krstic \([10]\) achieved a global asymptotic stability using Neumann and Dirichlet boundary control for the Burgers equation. The KdV equation has been intensively studied from various aspects of both mathematics and physics since the 1940s. In \([12, 14]\), solutions were discovered through solving directly the KdV equation. In \([11, 13, 15]\), the inverse scattering method was invented to seek solutions. In \([16]\), the author studied the exact boundary controllability of the KdV equation. The Korteweg de Vries-Burgers (KdVB) equation is one of the simplest nonlinear mathematical models displaying the features of both dispersion and dissipation. The study of the KdVB equation with periodic boundary conditions can be seen in \([17, 20]\). The case where the spatial domain is the whole real line can be seen in \([12]\). The controllability of the KdVB equation on a bounded domain can be seen in \([19]\), the stabilization on a bounded domain can be found in \([21]\). Liu and Krstic \([18]\) considered a boundary feedback stabilization problem of KdVB equation on a finite spatial interval. Andras Balogh and Krstic \([20]\) studied the stabilization and numerical demonstration of KdVB equation. bounded domain see Zhang \([21]\). Liu and Krstic \([18]\) considered a boundary feedback stabilization problem of KdVB equation on a finite spatial interval. Andras Balogh and Krstic \([20]\) studied the stabilization and numerical demonstration of KdVB equation. Kuramoto-Sivashinsky equation (K-S
equation) was derived independently by Kuramoto et al as a model for phase turbulence in reaction diffusion systems and by Sivashinsky as a model for plane flame propagation. So far, The K-S equation has been extensively studied, such as Foias et al [1] and Nicolaenko et al [2, 3, 4] who described the global attractors and inertial manifolds of the K-S equation. At the same time the control problem for the K-S equation is largely explored. For example, Wei-Jiu Liu and Miroslav Krstic [6] studied stability enhancement by boundary control in the K-S equation. Christofides [4] developed linear controllers based on a Galevkin truncation to achieve local stabilization, and so on.

There have been many studies on time periodic solutions of partial differential equations in the literature. For recently work on this subject, see Vejvoda et al [24], Keller and Ting [22], Robinowitz [23]. Wayne [25] gave a recent review on the periodic solutions of nonlinear partial differential equations. In this paper, we mainly use the Banach contraction fixed point theorem and semi-group theory to prove the uniqueness and existence of the solution. We prove the large-time behavior and stability of the solution, by usual truncation to achieve local stabilization, and so on.

In this paper, we will discuss stability of the following K-S equation with external excitation $f$ under the given boundary feedback conditions

$$u_t + u_{xxxx} + \lambda u_{xx} + uu_x = f$$  \hspace{1cm} (1)

The equation is subjected to the initial condition and the boundary conditions

$$u(x, 0) = \phi(x)$$  \hspace{1cm} (2)

$$u(0, t) = u(1, t) = u_x(0, t) = u_x(0, t) = 0$$  \hspace{1cm} (3)

$$u_{xx}(1, t) = -k u_x(1, t)$$  \hspace{1cm} (4)

where $0 < x < 1$, $t > 0$, $0 < \lambda < 1, k > 0$ and $f$ is a given external excitation function.

### 2 Definitions and notation

In what follows, let $H^s(0, 1)$ denote the usual Sobolev space, for any $s \in \mathbb{R}$. We denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the norm and scalar product of $L^2(0, 1)$ respectively. $L^2([0, T], L^2(0, 1))$ denotes the space of all functions $u(x, t)$ with $u(x, t) \in L^2(0, 1)$ for $x$ and $u(x, t) \in L^2[0, T]$ for $t$. If $u \in L^2([0, T], L^2(0, 1))$, the norm is defined by

$$\|u\|_{L^2([0, T], L^2(0, 1))} = \left( \int_0^T \int_0^1 u(x, t)^2 \, dx \, dt \right)^{1/2},$$

and it is simply denoted by $\|u\|_{L^2(L^2)}$. Likewise, $\|u\|_{L^2(0, 1)}$ is simply denoted by $\|u\|_{L^2}$ or $\|u\|_2$, $\|u\|_{L^2(R, H^{1}(0, 1))}$ by $\|u\|_{L^2(H^1)}$, $(i = 1, 2, 3, 4, 5)$, $\|u\|_{H^0(0, 1)}$ by $\|u\|_{H^0}$ $(i = 1, 2, 3, 4)$. Let $X$ be a Banach space and $T > 0$, we denote $C([0, T], X)$ for the space of all continuous functions defined on $[0, T]$ in space $X$.

We define the following new spaces. Let $X^j_T(j = 0, \cdots, 4)$ be the collection of all the following forms

$$\mathcal{A} = \left\{ (\phi, f) \in L^{j+2}(0, 1) \times H^{j} \left( [0, T] \times L^2(0, 1) \right) \cap L^2 \left( [0, T] \times H^{j+1}(0, 1) \right) \right\},$$

where $j = 0, \cdots, 4$ and $\phi$ satisfies the compatibility conditions $\phi(0) = \phi(1) = \phi'(0) = \phi''(0) = 0$. 

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Let $Y^j_T = C \left( [\tau, T], H^j (0, 1) \right) \cap L^2 \left( [\tau, T], H^{j+1} (0, 1) \right)$. And if $\tau = 0$, $Y^j_T$ is simply denoted by $Y^j_T$. We give the norm in the space as follows

$$
\| (\phi, f) \|_{X^j_T} = \left( \| \phi \|^2_{H^{j+2}([0, T])} + \| f \|^2_{H^j [0, T], L^2 (0, 1)} + \| f \|^2_{L^2 [0, T], H^{j+1} (0, 1)} \right)^{1/2}, \text{for } (\phi, f) \in X^j_T.
$$

$$
\| u \|_{Y^j_T} = \| u \|_{C([\tau, T], H^j (0, 1))} + \| u \|_{L^2([\tau, T], H^{j+1} (0, 1))}, \text{for } u \in Y^j_T.
$$

3 Main results

**Theorem 1.** Let $T > 0$. For any $(\phi, f) \in X^j_T$, $(j = 0, 4)$, with $\phi$ satisfying compatibility conditions $\phi (0) = \phi (1) = \phi'' (0) = \phi'' (1) = 0$, if $j = 4$, then the initial and boundary value problem (1)-(4) admits a unique solution $u \in Y^j_T$ satisfying

$$
\| u \|_{Y^j_T} \leq \gamma \left( \| (\phi, f) \|_{X^j_T} \right) \| (\phi, f) \|_{X^j_T}
$$

where $\gamma : R^+ \to R^+$ is a non-decreasing continuous function.

**Theorem 2.** If there exists $\eta \in (0, \frac{1}{2})$ and $f \in C \left( (R^+, L^2 (0, 1)) \cap L^2 \left( R^+, H^4 (0, 1) \right) \right)$ is a time periodic function with the period $\omega$ and satisfies

$$
\sup_{0 \leq t \leq \omega} \| f (\cdot, t) \|_{L^2 (0, 1)} < \eta
$$

then the system (1)-(4) admits a unique time periodic solution $u^* \in C \left( R^+, H^4 (0, 1) \right)$ with the period $\omega$ satisfying the boundary conditions, and there exists a $\delta > 0$, such that for any $\phi \in L^2 (0, 1)$, the solution $u (x, t)$ of the system (1)-(4) satisfies $\| u (\cdot, t) - u^* (\cdot, t) \|_{L^2} \leq Ce^{-\delta t}, \forall t \geq 0$.

4 Proof of Theorems

Firstly, we consider the homogeneous linear problem

$$
u_t + u_{xxxx} + \lambda u_{xx} = 0, u (x, 0) = \phi (x)
$$

$$
u (0, t) = u (1, t) = u_x (0, t) = 0, u_{xx} (0, t) = 0, u_{xxx} (1, t) = -ku_x (1, t)
$$

(7)

where $0 < x < 1, t > 0, 0 < \lambda < 1, k$ are the same as the before defining.

Let $W (t)$ be the $C_0$semi-group generated by the operator $A \psi = -\psi'''' - \lambda \psi''$ from $L^2 (0, 1)$ to $L^2 (0, 1)$ with $D (A) = \{ \psi \in H^4 (0, 1), \psi (0) = \psi (1) = \psi' (0) = \psi'' (0) = 0 \}$.

Then the solution of the homogeneous linear problem (7) is given by $u (t) = W (t) \phi$. By d’Alembert formula, we can use the semi-group $W (t)$ to formally write the solution of the inhomogeneous linear problem

$$
u_t + u_{xxxx} + \lambda u_{xx} = f, u (x, 0) = \phi (x),
$$

$$
u (0, t) = u (1, t) = u_x (0, t) = u_{xx} (0, t) = 0, u_{xxx} (1, t) = -ku_x (1, t)
$$

(8)

where $0 < x < 1, t > 0, 0 < \lambda < 1, k$ are the same as the before defining. Then its solution is given by

$$
u_t = \int_0^T W (t - \tau) f (\cdot, \tau) d\tau, \forall t > \tau \geq 0.
$$

In the following, we give two lemmas which will play an important role later in studying stability of time periodic solution of the nonlinear system (1)-(4). In what follows, $C$ is a kind of positive constants.

**Lemma 1.** Let $T > 0$ be given and $u$ be a solution of (7). Then there exists a constant $C$ (independent of $\phi$) such that

$$
\| u \|_{V^j_T} \leq C \| \phi \|_{H^1}
$$

(9)
if $\phi \in H^5(0, 1)$, then
\[ \|u\|_{Y^d_T} + \|u_t\|_{Y^d_T} \leq C \|\phi\|_{H^5} \tag{10} \]

**Lemma 2.** Let $T > 0$ be given and $u$ be a solution of (8), then there exists a constant $C$ (independent of $f$) such that

if $f \in L^2(H^1)$
\[ \|u\|_{Y^d_T} \leq C \|f\|_{L^2(H^1)} \tag{11} \]

if $f \in L^2(H^3) \cap H^1(H^2)$,
\[ \|u\|_{Y^d_T} + \|u_t\|_{Y^d_T} \leq C \left( \|f\|_{L^2(H^3)} + \|f_t\|_{L^2(H^2)} \right) \tag{12} \]

We only present a proof for Lemma 1. The proof of Lemma 2 is similar to Lemma 1.

**Proof of Lemma 1:** Multiplying both sides of the equation in (7) by $2u$ and integrating over $(0,1)$ with respect to $x$, we have
\[ \frac{d}{dt} \int_0^1 u^2(x, t) \, dx + 2u u_{xxx} \|u\|_{H^1}^2 + 2u_x u_{xx} \|u\|_{H^3}^2 + 2\lambda \int_0^1 u_x^2(x, t) \, dx = 0 \]

Since $2 \int_0^1 u^2(x, t) \, dx \leq \int_0^1 u_x^2(x, t) \, dx$, then from the above conditions, one can easily deduce $\|u\|_{Y^d_T} \leq C \|\phi\|_{H^1}$.

In order to prove (10), let $v = u_t$. Then $v$ satisfies
\[ v_t + v_{xxx} + \lambda v_{xx} = 0 \quad v(x, 0) = \phi^s(x), \]
\[ v(0, t) = v(1, t) = v_x(0, t) = v_{xx}(0, t), \quad v_{xx}(1, t) = -kv_x(1, t) \]
where $\phi^s(x) = -\phi'''(0) \lambda \phi''(0)$, and $0 < x < 1, t > 0, 0 < \lambda < 1, k$ are the same as the before defining. By (9), we have
\[ \|u_x\|_{L^2} \leq C \|\phi\|_{H^1} \leq C \|\phi\|_{H^3}, \quad \|u_{xx}\|_{L^2} \leq C \|\phi\|_{H^1} \leq C \|\phi\|_{H^5} \]
\[ \|u_t\|_{L^2} \leq C \|\phi\|_{H^1} \leq C \|\phi\|_{H^5}, \quad \|u_{xxx}\|_{L^2} \leq C \|\phi\|_{L^2(H^1)} \leq C \|\phi\|_{H^5}. \]

By $u_{xxx} = -u_t - \lambda u_{xx}$, we obtain
\[ \|u_{xxx}\|_{L^2} \leq C \|u_{xxxx}\|_{L^2(H^1)} \]
\[ \|u_{xxx}\|_{L^2} \leq C \left( \|u_t\|_{L^2} + \lambda \|u_x\|_{L^2} \right) \]
and $\sup_{0 \leq t \leq T} (\|u\|_{Y^d_T}) \leq C \|\phi\|_{H^5}$. Then this is (10).

Obviously there exists the $f$ satisfying the conditions of the Theorem 1. Now we prove Theorem 1.

**Proof of Theorem 1:** Using the notation of the semi-group $W(t)$, rewrite (1) in its integral form
\[ u(t) = W(t) \phi + \int_0^t W(t - \tau) f(\cdot, \tau) \, d\tau - \int_0^t W(t - \tau) (u u_x)(\cdot, \tau) \, d\tau. \]

Let $r > 0, \theta > 0$ be two constants, $S$ denotes the following set $S = \left\{ v \in Y^0_{T}; \|v\|_{Y^0_T} \leq r \right\}$. For given $r$ and $\theta$, $S$ is a complete metric space. For $(\phi, f) \in X_{H}^{0}$ and any $v \in S$, define a map $\Gamma$ on $S$ as follows
\[ \Gamma(v) = W(t) \phi + \int_0^t W(t - \tau) f(\cdot, \tau) \, d\tau - \int_0^t W(t - \tau) (v v_x)(\cdot, \tau) \, d\tau. \]

By Lemma 1 and Lemma 2, we have
\[ \|\Gamma(v)\|_{Y^0_T} \leq C \|\phi, f\|_{X_{T}^{0}} + C \int_0^\theta \|v v_x(\cdot, \tau)\|_{L^2} \, d\tau \]
\[ \leq C \|\phi, f\|_{X_{T}^{0}} + C \int_0^\theta \|v(\cdot, \tau)\|_{L^\infty} \|v_x(\cdot, \tau)\|_{L^2} \, d\tau. \]
\[
\leq C \|\phi, f\|_{X_0^r} + C \int_0^\theta \|v\|_{L^2}^{1/2} \|v_x\|_{L^2}^{1/2} \|v_x\|_{L^2} \, d\tau
\]
\[
\leq C \|\phi, f\|_{X_0^r} + C \sup_{0 \leq \tau \leq \theta} \|v(\cdot, \tau)\|_{L^2}^{1/2} \int_0^\theta \|v_x(\cdot, \tau)\|_{L^2}^{3/2} \, d\tau
\]
\[
\leq C \|\phi, f\|_{X_0^r} + C \|v\|_{Y_0^r}^2
\]

If we take \( r = 2C \|\phi, f\|_{X_0^r} \) and \( C \theta r \leq \frac{1}{2} \), then \( \|\Gamma(v)\|_{Y_0^r} \leq r \) for any \( v \in S \). So \( \Gamma \) maps \( S \) into \( S \). In the same way, we can prove that for \( r \) and \( \theta \) chosen as the above, we have
\[
\|\Gamma(v_1) - \Gamma(v_2)\|_{Y_0^r} \leq \frac{1}{2} \|v_1 - v_2\|_{Y_0^r}
\]

In other words, the map \( \Gamma \) is a contraction. Hence, by the Banach contraction fixed point theorem, \( \Gamma \) has a unique fixed point \( u = \Gamma(u) \) which is the unique solution of (1)-(4) in the space \( S \). Thus we obtain the local solution of (1)-(4) in the space \( X_0^r \), in order to obtain the global well-posedness, we must prove estimate (5).

As follows, we will prove that estimate (5) with \( j = 0 \) holds for any smooth solution \( u \) of (1)-(4).

Multiplying both sides of the equation (1) by \( 2u \) and then integrating from 0 to 1 by parts, we obtain
\[
\frac{d}{dt} \int_0^1 u^2 \, dx - 2u_x(1, t)u_x(1, t) + 2 \int_0^1 u_{xx}^2 \, dx - 2\lambda \int_0^1 u_x^2 \, dx = 2 \int_0^1 uf \, dx
\]

Since
\[
\frac{d}{dt} \int_0^1 u^2 \, dx + \int_0^1 u_{xx}^2 \, dx + \int_0^1 u^2 \, dx \leq 2 \int_0^1 u^2 \, dx + \int_0^1 u_x^2 \, dx + \int_0^1 f^2 \, dx,
\]
by Lemma 1 and Lemma 2, we have
\[
\|u\|_{Y_0^r} \leq \gamma \left( \|\phi, f\|_{X_0^r}\right) \|\phi, f\|_{X_0^r}.
\]

For \( (\phi, f) \in X_0^r \), let \( S^* \) denote the set \((r \) and \( \theta \) are chosen as the above): \( S^* = \{v \in Y_0^r; v_t \in Y_0^r, \|v\|_{Y_0^r} + \|v_t\|_{Y_0^r} \leq r\} \)
In the same way, firstly we establish \( S^* \rightarrow S^* \), when \( r \) and \( \theta \) are chosen properly, it can show that \( \Gamma \) is a contraction mapping from \( S^* \) to \( S^* \). Thus (1)-(4) is locally well-posed in the space \( X_0^r \).

To obtain the global well-posedness, we must show the estimate (5). Let \( h = u_t \), then
\[
h_t + h_{xxxx} + \lambda h_{xx} + (uh)_x = f_t, h(x, 0) = \phi^*(x),
\]
\[
h(0, t) = h(1, t) = h_x(0, t) = h_x(1, t) = 0
\]
\[
h_{xx}(1, t) = -\lambda h_x(1, t),
\]
where \( \phi^*(x) = f(x, 0) - \phi'''(x) - \lambda \phi''(x) - \phi(x) \phi'(x), 0 < x < 1, t > 0, 0 < \lambda < 1, k \) are the same as the before defining. Repeating the above procedure, we have
\[
\frac{d}{dt} \int_0^1 h^2 \, dx + 2 \int_0^1 h_{xxxx} \, dx + 2\lambda \int_0^1 h_{xx} \, dx + 2 \int_0^1 h(uh)_x \, dx = 2 \int_0^1 h f_t \, dx
\]
then
\[
\frac{d}{dt} \int_0^1 h^2 \, dx \leq \int_0^1 h^2 \, dx + \int_0^1 f_t^2 \, dx + 2 \|u\|_{L^2} \int_0^1 h_x^2 \, dx
\]
and
\[
\frac{d}{dt} \int_0^1 h^2 \, dx + \int_0^1 h^2 \, dx + \int_0^1 h_{xx}^2 \, dx \leq 2 \int_0^1 h^2 \, dx + \int_0^1 f_x^2 \, dx + (2 \|u\|_{L^2} + 1) \int_0^1 h_x^2 \, dx.
\]
So we have
\[
\|h\|_{L^2} \leq C \left( \|h\|_{L^2(L^2)} + \|f_t\|_{L^2(L^2)} + \|h_x\|_{L^2(L^2)} \right)
\]

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\[ C \left( \| \phi \|_{L^2(H^2)} + \| f \|_{L^2(H^1)} \right) \leq C \left( \| u \|_{Y_T^0} \right) \leq C \left( \| \phi, f \|_{X_T^1} \right). \]

Since

\[ \| u \|_{L^2} \leq C \| \phi \|_{H^1}, \quad \| u_x \|_{L^2} \leq C \| \phi \|_{H^1}, \]

\[ \| u_{xx} \|_{L^2(L^2)} \leq C \| u_x \|_{L^2(H^1)} \leq C \| u \|_{L^2(H^2)}, \]

\[ \| u_{xxxx} \|_{L^2(L^2)} \leq C \| u_{xx} \|_{L^2(H^1)} \leq C \left( \| u_t \|_{L^2(L^2)} + \| u_x \|_{L^2(L^2)} + \| u_{xx} \|_{L^2(L^2)} + \| f \|_{L^2(L^2)} \right) \]

and

\[ \| u_x \|_{L^2(L^2)} = \left( \int_0^T \| u_x \|_{L^2}^2 \, dt \right)^{1/2} \leq \left( \int_0^T \max_{x(0,1)} u(x,t) \| u_x \|_{L^2} \, dt \right)^{1/2} \]

\[ \leq \left( \max_{x(0,1), t(0,T)} u(x,t) \right) \| u_x \|_{L^2(L^2)} \leq C \| u \|_{L^2(L^2)} \]

then we have

\[ \| u_{xxx} \|_{L^2(L^2)} \leq C \left( \| u_t \|_{L^2(L^2)} + \| u_x \|_{L^2(L^2)} + \| f \|_{L^2(L^2)} \right), \]

\[ \| u_{xxxx} \|_{L^2(L^2)} \leq C \left( \| u_t \|_{L^2(L^2)} + \| u_x \|_{L^2(L^2)} + \| u_{xx} \|_{L^2(L^2)} + \| f \|_{L^2(L^2)} \right), \]

\[ \| u_{xxxxx} \|_{L^2(L^2)} \leq C \| u_{xx} - \lambda u_{xxx} - u_{xx} \|_{L^2(L^2)} \]

\[ \leq C \left( \| \phi \|_{H^6} + \| f \|_{L^2(H^1)} + \| u_{xx} \|_{L^2(H^1)} + \| u \|_{L^2(L^2)} + \| f \|_{L^2(H^1)} \right). \]

All in all, we obtain

\[ \| u \|_{Y_T^0} \leq \gamma \left( \| (\phi, f) \|_{X_T^1} \right) \| (\phi, f) \|_{X_T^1}. \]

Now in the following we will prove the large-time behavior of the solution of the system (1)-(4).

**Lemma 3.** Let \( T > 0, 0 < \varepsilon < 1 \) be given, for \( f \in L^2 \left( R^+; L^2(0,1) \right) \) and \( \phi \in L^2(0,1) \), the solution \( u \) of (1) satisfies

\[ \| u(\cdot, \tau) \|_{L^2} \leq e^{-(1-\varepsilon)\tau} \| \phi \|_{L^2} + \sqrt{\frac{1}{2\varepsilon}} \left\| \int_0^T e^{-2(1-\varepsilon)(t-\tau)} \| f(\cdot, \tau) \|_{L^2}^2 \, d\tau \right\|_{L^2} \]

\[ + \sqrt{\frac{1}{2\varepsilon}} \left\| \int_s^t e^{-2(1-\varepsilon)(t-\tau)} \| f(\cdot, \tau) \|_{L^2}^2 \, d\tau \right\|_{L^2} \]

and

\[ \| u \|_{L^2(t,t+T; H^2(0,1))} \leq \sqrt{2 + \frac{1}{4(1-\lambda)}} \| f \|_{L^2(t,t+T; L^2(0,1))}, \quad 0 \leq s \leq t < \infty. \]

Consequently, assuming \( f \in L^2 \left( R^+, L^2(0,1) \right) \),

(i) if \( \lim_{t \to \infty} \| f(\cdot, t) \|_{L^2} = 0 \), then

\[ \lim_{t \to \infty} \left( \| u(\cdot, t) \|_{L^2(0,1)} + \| u \|_{L^2(t,t+T; H^2(0,1))} \right) = 0 \]

(ii) if \( \| f(\cdot, t) \|_{L^2} \leq Ce^{-\alpha t}, \quad (\alpha > 0) \), then

\[ \| u(\cdot, t) \|_{L^2} + \| u \|_{L^2(t,t+T; H^2(0,1))} \leq Ce^{-(1-\varepsilon)t} \| \phi \|_{L^2} + \sqrt{2 + \frac{1}{4(1-\lambda)}} Ce^{-(1-\varepsilon)t} \]

\[ \geq e^{-\alpha t} \cdot \min \{ 1, \alpha \} t \]

\[ \| u(\cdot, t) \|_{L^2} + \| u \|_{L^2(t,t+T; H^2(0,1))} \leq Ce^{-(1-\varepsilon)t} \| \phi \|_{L^2} + \sqrt{2 + \frac{1}{4(1-\lambda)}} Ce^{-(1-\varepsilon)t} \]

\[ \geq e^{-\alpha t} \cdot \min \{ 1, \alpha \} t \]
where $C_{\varepsilon, \alpha} = \begin{cases} \sqrt{\frac{1}{\varepsilon} - \alpha} + \sqrt{\frac{1}{2\alpha}}, & \text{if } \alpha \neq 1 - \varepsilon \\ \varepsilon + \sqrt{\frac{1}{2\alpha}}, & \text{if } \alpha = 1 - \varepsilon \end{cases}$, $t \geq 0$.

**Remark:** Obviously there exists $f$ satisfying the conditions of Lemma 3. For example $f = xe^{-2\alpha t}$ ($\alpha > 0$ the same as in Lemma 3), then $f \in L^2 \left( R^+; L^2 (0, 1) \right)$. And because of $\|f\|_{L^2} = e^{-2\alpha t}$, the conditions $\lim_{t \to \infty} \|f(\cdot, t)\|_{L^2} = 0$ and $\|f(\cdot, t)\|_{L^2} \leq C e^{-\alpha t}$, ($\alpha > 0$) exist. Hence the Lemma is meaningful.

**Proof:** For given $\phi$ and $f$, the solution of the system (1)-(4) satisfies the following equality
\[ \frac{d}{dt} \int_0^1 u^2 dx - 2u_x(1, t) u_x(1, t) + 2 \int_0^1 u_x^2 dx - 2\lambda \int_0^1 u^2 dx = 2 \int_0^1 f u dx \]
and since $2 \int_0^1 u^2 dx = \int_0^1 u^2 dx$ then
\[ \frac{d}{dt} \int_0^1 u^2 dx + 2k u_x(1, t)^2 + 2 \int_0^1 u_x^2 dx - 2\lambda \int_0^1 u^2 dx \leq 2\varepsilon \|u\|^2_{L^2} + \frac{1}{2\varepsilon} \|f\|^2_{L^2} \]
where $0 < \varepsilon < 1$. We obtain
\[ \frac{d}{dt} \int_0^1 u^2 dx \leq -2 \|u_x\|^2 + 2\lambda \|u_x\|^2 + 2\varepsilon \|u\|^2 + \frac{1}{2} \|f\|^2 \]
\[ \leq -4 \|u_x\|^2 + 2\lambda \|u_x\|^2 + 2\varepsilon \|u\|^2 + \frac{1}{2} \|f\|^2 \]
by Gronwall inequality. Thus we have
\[ \int_0^1 u^2 dx \leq e^{-2(1-k)(t-t)} \|\phi\|^2_{L^2} + \frac{1}{2\varepsilon} \int_0^t e^{-2(1-k)(\tau-t)} \|f(\cdot, \tau)\|^2_{L^2} \, d\tau, \]
$(t > 0)$. In particular, for any $0 \leq s \leq t$, we have
\[ \int_0^s u^2 dx \leq e^{-2(1-k)(t-s)} \|\phi(s)\|^2_{L^2} + \frac{1}{2\varepsilon} \int_0^t e^{-2(1-k)(\tau-t)} \|f(\cdot, \tau)\|^2_{L^2} \, d\tau \]
\[ \leq e^{-2(1-k)(t-s)} \left( e^{-2(1-k)s} \|\phi\|^2_{L^2} + \frac{1}{2\varepsilon} \int_0^s e^{-2(1-k)(\tau-t)} \|f(\cdot, \tau)\|^2_{L^2} \, d\tau \right) + \frac{1}{2\varepsilon} \int_0^t e^{-2(1-k)(\tau-t)} \|f(\cdot, \tau)\|^2_{L^2} \, d\tau \]
This is (13).

As followings, we will prove (14). By the above proof, we have
\[ \frac{d}{dt} \int_0^1 u^2 dx \leq -4 \|u_x\|^2 + 2\lambda \|u_x\|^2 + \frac{1}{4} \|u\|^2 + \|f\|^2 \]
Then
\[ \int_t^{t+T} \|u_x(\cdot, \tau)\|^2_{L^2} \, d\tau + \|u\|^2_{L^2} \leq \int_t^{t+T} \|f(\cdot, \tau)\|^2_{L^2} \, d\tau. \]
In the same way, we have
\[ \frac{d}{dt} \int_0^1 u^2 dx \leq -2 \|u_x\|^2 + 2\lambda \|u_x\|^2 + 4(1 - \lambda) \|u\|^2 + \frac{1}{4(1-\lambda)} \|f\|^2 \]
Then
\[ \int_t^{t+T} \|u(\cdot, \tau)\|^2_{L^2} \, d\tau + \|u\|^2_{L^2} \leq \frac{1}{4(1-\lambda)} \int_t^{t+T} \|f(\cdot, \tau)\|^2_{L^2} \, d\tau. \]
It follows easily that
\[ \|u\|^2_{L^2(t,t+T; H^2(0,1))} \leq \sqrt{2 + \frac{1}{4(1-\lambda)} \|f\|^2_{L^2(t,t+T; L^2(0,1))}}. \]
This is (14). And (15) and (16) follow easily.

**Lemma 4.** Let $T > 0$, suppose

\[ f \in C \left( R^+; L^2 (0, 1) \right) \cap L^2 \left( R^+, H^1 (0, 1) \right) \cap H^1 \left( R^+, L^2 (0, 1) \right). \]
If $f$ satisfies the condition $\lim_{t \to +\infty} \|f(\cdot, t)\|_{L^2(0,1)} < \frac{1}{2}$ and $\phi \in H^4 (0,1)$ satisfies the boundary conditions of (1)-(4), then for any $0 < \eta < 1 - 2 \lim_{t \to +\infty} \|f(\cdot, t)\|_{L^2}$, there exists $s_1 > 0$ and $\|\phi\|_{L^2} + \|f\|_{L^2(L^2)}$, such that the solution of (1)-(4) satisfies

\[
\|u\|_{H^4} \leq \gamma \left( \|\phi\|_{L^2} + \|f\|_{L^2(L^2)} \right) \left\{ \|u(\cdot, s_1)\|_{L^2} + \|u_t(\cdot, s_1)\|_{L^2} \right\} e^{-\eta(t-s_1)}
\]

\[
+ \left[ \int_s^t e^{-2\eta(t-\tau)} \left( \|f(\cdot, \tau)\|_{L^2}^2 + \|f_t(\cdot, \tau)\|_{L^2}^2 \right) d\tau \right]^{1/2}
\]

\[
+ e^{-\eta(t-s)} \left[ \int_{s_1}^t e^{-2\eta(s-\tau)} \left( \|f(\cdot, \tau)\|_{L^2}^2 + \|f_t(\cdot, \tau)\|_{L^2}^2 \right) d\tau \right]^{1/2} + \|f\|_{L^2(L^2)} + \|f\|_{H^1(L^2)}
\]

Moreover

(i) If $\lim_{t \to +\infty} \left( \|f(\cdot, t)\|_{L^2} + \|f_t(\cdot, t)\|_{L^2} + \|f\|_{L^2(t,t+T;H^1(0,1))} + \|f\|_{H^1(L^2)} \right) = 0$

Then

\[
\lim_{t \to +\infty} \|u\|_{Y_{t+T}} = 0
\]

(ii) If $\|f(\cdot, t)\|_{L^2} + \|f_t(\cdot, t)\|_{L^2} + \|f\|_{L^2(H^1)} + \|f\|_{H^1(L^2)} < Ce^{-at}$, $(a > 0, t \geq 0)$.

Then

\[
\|u\|_{Y_{t+T}} \leq \gamma \left( \|\phi\|_{L^2} + \|f\|_{L^2(L^2)} \right) \left\{ 2 \|u(\cdot, s_1)\|_{L^2} + \|u_t(\cdot, s_1)\|_{L^2} \right\} e^{-\eta(t-s_1)}
\]

\[
+ C(2\sqrt{t-s} + 2\sqrt{s-s_1} + 3) e^{-\min(\eta, \alpha)t} + Ce^{-at}
\]

Remark: Obviously there exists $f$ satisfying the conditions of Lemma 4, for example we can take $f = \frac{1}{2} xe^{-2at}$, then $f$ satisfies $f \in C \left( R^+, L^2(0,1) \right) \cap L^2 \left( R^+, H^1 (0,1) \right) \cap H^1 \left( R^+, L^2 (0,1) \right)$ and $\|f\|_{L^2} = \frac{1}{2} e^{-2at}$, thus we have $\lim_{t \to +\infty} \|f(\cdot, t)\|_{L^2(0,1)} < \frac{1}{2}$. Meanwhile

\[
\lim_{t \to +\infty} \left( \|f(\cdot, t)\|_{L^2} + \|f_t(\cdot, t)\|_{L^2} + \|f\|_{L^2(t,t+T;H^1(0,1))} + \|f\|_{H^1(L^2)} \right) = 0
\]

and $\|f(\cdot, t)\|_{L^2} + \|f_t(\cdot, t)\|_{L^2} + \|f\|_{H^1(L^2)} \leq Ce^{-at}$, $(a > 0, t \geq 0)$ must be exist. So Lemma 4 is meaningful.

Proof of Lemma 4: Let $h = u_t$. Then we have

\[
h_t + h_{xxxx} + \lambda h_{xx} + (uh)_x = f_t
\]

\[
h(x, 0) = \phi^*(x)
\]

\[
h(0, t) = h(1, t) = h_{xx}(0, t) = h_{xx}(0, t)
\]

\[
h_{xx}(1, t) = -kh_{xx}(1, t)
\]

where $\phi^*(x) = f(x, 0) - \phi''''(x) - \lambda \phi''(x) - \phi(x) \phi'(x), 0 < x < 1, t > 0, 0 < \lambda < 1, k$ are the same as the before define. Then obviously in the same way, we have

\[
\frac{d}{dt} \int_0^1 h^2 dx + 2kh(1, t)^2 + 2 \int_0^1 h_x^2 dx - 2\lambda \int_0^1 h_x^2 dx = 2 \int_0^1 uh_x dx + 2 \int_0^1 h f_t dx
\]
by Gronwall inequality
\[
\|h(\cdot, t)\|_{L^2}^2 \leq e^{-2\eta(s-t)} \|h(\cdot, s)\|_{L^2}^2 + \frac{1}{2\varepsilon} \int_s^t e^{-2\eta(s-t\tau)} \|f_t(\cdot, \tau)\|_{L^2}^2 d\tau
\]
and
\[
\int_t^{t+T} \|h_x(\cdot, \tau)\|_{L^2}^2 d\tau \leq \frac{1}{4e \eta(s,t)} \int_t^{t+T} \|f_t(\cdot, \tau)\|_{L^2}^2 d\tau + \frac{1}{2\eta(s,t)} \|h(\cdot, t)\|_{L^2}^2
\]
where \(\eta(s,t) = 1 - \varepsilon - \lambda - \sup \|u(\cdot, t)\|_{L^2}, 0 \leq s \leq t, 0 < \lambda < 1.\) Moreover since \(\lim_{t \to +\infty} \|u(\cdot, t)\|_{L^2} \leq 2 \lim_{t \to +\infty} \|f(\cdot, t)\|_{L^2} < 1\)

for given \(0 < \eta < 1 - 2 \lim_{t \to +\infty} \|f(\cdot, t)\|_{L^2},\) let \(\varepsilon = \frac{1}{2} \left(1 - \eta - 2 \lim_{t \to +\infty} \|f(\cdot, t)\|_{L^2}\right) - \lambda,\) then by Lemma 3, there exists \(s_1 > 0\) (independent of \(\|\phi\|_{L^2} + \|f\|_{C(\mathbb{R}^+; L^2)}\)), such that for any \(s > s_1,\) we have
\[
\|h(\cdot, t)\|_{L^2} \leq e^{-\eta(t-s)} \|h(\cdot, s)\|_{L^2} + \frac{1}{2\varepsilon} \left(\int_s^t e^{-2\eta(t-s\tau)} \|f_t(\cdot, \tau)\|_{L^2}^2 d\tau\right)^{1/2}
\]
(22)

In particular, we have
\[
\|h(\cdot, s)\|_{L^2} \leq e^{-\eta(s-s_1)} \|h(\cdot, s_1)\|_{L^2} + \frac{1}{2\varepsilon} \left(\int_{s_1}^s e^{-2\eta(s-s\tau)} \|f_t(\cdot, \tau)\|_{L^2}^2 d\tau\right)^{1/2} \quad (s > s_1)
\]
(23)
and
\[
\|h(\cdot, t)\|_{L^2} \leq e^{-\eta(t-s_1)} \|h(\cdot, s_1)\|_{L^2} + \frac{1}{2\varepsilon} e^{-\eta(t-s_1)} \left(\int_{s_1}^s e^{-2\eta(s-s\tau)} \|f_t(\cdot, \tau)\|_{L^2}^2 d\tau\right)^{1/2}
\]
\[
+ \frac{1}{2\varepsilon} \left(\int_{s_1}^s e^{-2\eta(s-s\tau)} \|f_t(\cdot, \tau)\|_{L^2}^2 d\tau\right)^{1/2}, \quad s_1 < s \leq t.
\]
(24)

By Theorem 1, there exists \(\gamma = \gamma \left(\|\phi, f\|_{X_0^4}\right)\) such that
\[
\|h(\cdot, s_1)\|_{L^2} \leq \gamma \left(\|\phi, f\|_{X_0^4}\right) \|\phi, f\|_{X_0^4_1}.
\]

Then
\[
\|u_t(\cdot, t)\|_{L^2} \leq \gamma \left(\|\phi, f\|_{X_0^4_1}\right) \|\phi, f\|_{X_0^4_1} e^{-\eta t} + \frac{1}{2\varepsilon} e^{-\eta(t-s_1)} \left(\int_{s_1}^s e^{-2\eta(s-s\tau)} \|f_t(\cdot, \tau)\|_{L^2}^2 d\tau\right)^{1/2}
\]
\[
+ \frac{1}{2\varepsilon} \left(\int_{s_1}^s e^{-2\eta(s-s\tau)} \|f_t(\cdot, \tau)\|_{L^2}^2 d\tau\right)^{1/2}
\]
and with (21), we obtain
\[
\|u_t\|_{L^2(L^2(0,t+2H^2(0,1)))} \leq 3 \sqrt{\frac{1}{2\eta(s,t)}} \|u_t(\cdot, t)\|_{L^2} + 6 \sqrt{\frac{1}{\gamma s_1(t)}} \|f_t\|_{L^2(L^2)} (s_1 < s \leq t).
\]

Since \(\|u_x\|_{L^2} \leq C \|f\|_{L^2},\) \(\|u_{xx}\|_{L^2} \leq C \|f\|_{L^2}\)
and
\[
u_{xxxx} = f - u_t - \lambda u_{xx} - uu_x
\]
(25)
then there exists a constant $C$, such that

\[
\|u_{xxx}(\cdot, t)\|_{L^2} \leq C \|u_{xxx}(\cdot, t)\|_{H^{-1}(0,1)} \leq C \left( \|f(\cdot, t)\|_{L^2} + \|u(t)\|_{L^2} + \|u^2(\cdot, t)\|_{L^2} + \|u_x(\cdot, t)\|_{L^2} \right)
\]

\[
\leq C \|f(\cdot, t)\|_{L^2} + \|u(t)\|_{L^2} + \|u^2(\cdot, t)\|_{L^2} + (1 + \|u(\cdot, t)\|_{L^2}) \|u_x(\cdot, t)\|_{L^2}.
\]

Then for given $T > 0$, we have

\[
\|u_{xxx}(\cdot, t)\|_{L^2(t,t+T;L^2(0,1))} \leq C \left[ \|f(\cdot, t)\|_{L^2(t,t+T;L^2(0,1))} + \|u(t)\|_{L^2(t,t+T;L^2(0,1))} + \sup_{t \leq \tau \leq \xi + T} (1 + \|u(\cdot, \tau)\|_{L^2}) \|u_x(\cdot, t)\|_{L^2(t,t+T;L^2(0,1))} \right].
\]

By (25), we obtain

\[
\|u_{xxx}\|_{L^2(L^2)} \leq \|f - u_t\|_{L^2(L^2)} + \| - \lambda u_{xx} - u_{xx}\|_{L^2(L^2)}
\]

\[
\leq \|f - u_t\|_{L^2(L^2)} + \| u_{xx}\|_{L^2(L^2)} + \| u_{xx}\|_{L^2(L^2)}
\]

\[
\leq \left( \|f\|_{L^2(L^2)} + \| u_t\|_{L^2(L^2)} \right) + \sup_{t \leq \tau \leq \xi + T} (1 + \|u(\cdot, \tau)\|_{L^2}) \|u_{xx}\|_{L^2(L^2)}.
\]

By Lemma 3, $\|u_x\|_{L^2(L^2)} \leq C \|f\|_{L^2(L^2)}$, $\|u_{xx}\|_{L^2(L^2)} \leq C \|f\|_{L^2(L^2)}$, then have

\[
\|u_{xxx}\|_{L^2(L^2)} \leq \|f\|_{L^2(L^2)} + \| u_t\|_{L^2(L^2)} + \sup_{t \leq \tau \leq \xi + T} (1 + \|u(\cdot, \tau)\|_{L^2}) \|f\|_{L^2(L^2)}
\]

\[
\leq C \sup_{t \leq \tau \leq \xi + T} (1 + \|u(\cdot, \tau)\|_{L^2}) \left( \|f\|_{L^2(L^2)} + \| u_t\|_{L^2(L^2)} \right).
\]

Since

\[
u \in L^2 \left(t, t + T; H^4(0,1)\right), \quad u_t \in L^2 \left(t, t + T; H^2(0,1)\right),
\]

it yields that

\[
u \in C(t, t + T; H^3(0,1))
\]

by interpolation and

\[
\sup_{t \leq \tau \leq \xi + T} \|u_{xxx}(\cdot, \tau)\|_{L^2} \leq C(\|u\|_{L^2(H^4)} + \| u_t\|_{L^2(H^2)}),
\]

where $C$ is independent of $t, T, u$. Likewise, from (25), we obtain

\[
\|u_{xxx}\|_{L^2(L^2)} \leq \|f - u_t\|_{L^2(L^2)} + \| - \lambda u_{xx} - u_{xx}\|_{L^2(L^2)}
\]

\[
\leq \|f - u_t\|_{L^2(L^2)} + (1 + \|u(\cdot, t)\|_{L^2}) \|u_{xx}(\cdot, t)\|_{L^2(L^2)}
\]

\[
\leq \|f - u_t\|_{L^2(L^2)} + C(1 + \|u(\cdot, t)\|_{L^2}) \|u(\cdot, t)\|_{L^2(H^2)} + \|u_t\|_{L^2(H^2)}
\]

\[
\leq \|f\|_{L^2(L^2)} + \|u_t\|_{L^2(L^2)} + C(1 + \|u(\cdot, t)\|_{L^2}) \|f\|_{L^2(L^2)} + \|u_t\|_{L^2(H^2)}
\]

\[
\leq \|f\|_{L^2(L^2)} + \|u_t\|_{L^2(L^2)} + C(1 + \|u(\cdot, t)\|_{L^2}) \|f\|_{L^2(L^2)} + \|u_t\|_{L^2(H^2)}
\]

and

\[
\|u_{xxxx}\|_{L^2(L^2)} \leq \|f - u_t\|_{L^2(H^1)} + 2 \|u_x\|_{L^2(L^2)} \sup_{t \leq \tau \leq \xi + T} \|u_{xx}\|_{L^2(L^2)}
\]

\[
\leq \|f\|_{L^2(L^2)} + \|u_x\|_{L^2(L^2)} + C(1 + \|u\|_{L^2(L^2)})(\|u\|_{L^2(H^3)} + \|u_t\|_{L^2(H^1)}).
\]

Repeating the above procedure, we can obtain the following results (22)---(24).

\[
\|u(\cdot, t)\|_{L^2} \leq e^{-\eta(t-s)} \|u(\cdot, s)\|_{L^2} + \sqrt{\frac{1}{2\epsilon}} \int_{t}^{\xi} e^{-\eta(\cdot, \tau)} \|f(\cdot, \tau)\|_{L^2(L^2)}^2 d\tau)^{1/2}
\]

\[
\|u(\cdot, s)\|_{L^2} \leq e^{-\eta(s-s_1)} \|u(\cdot, s_1)\|_{L^2} + \sqrt{\frac{1}{2\epsilon}} \int_{s_1}^{s} e^{-\eta(\cdot, \tau)} \|f(\cdot, \tau)\|_{L^2(L^2)}^2 d\tau)^{1/2} \quad (s > s_1)
\]

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By Lemma 3, (17), (19) follow easily. With the above definition, we have

\[
\|u(\cdot, t)\|_{L^2} \leq e^{-\eta(t-s_1)} \|u(\cdot, s_1)\|_{L^2} + \sqrt{\frac{1}{2\varepsilon}} e^{-\eta(t-s_1)} \left( \int_{s_1}^{t} e^{-2\eta(s-\tau)} \|f(\cdot, \tau)\|_{L^2}^2 \, d\tau \right)^{1/2} \\
+ \sqrt{\frac{1}{2\varepsilon}} \left( \int_{s_1}^{t} e^{-2\eta(s-\tau)} \|f(\cdot, \tau)\|_{L^2}^2 \, d\tau \right)^{1/2}, \quad s_1 < s \leq t.
\]

Hence we have

\[
\|u\|_{H^4} = (\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2 + \|u_{xxx}\|_{L^2}^2 + \|u_{xxxx}\|_{L^2}^2)^{1/2} \\
\leq \|u\|_{L^2} + \|u_x\|_{L^2} + \|u_{xx}\|_{L^2} + \|u_{xxx}\|_{L^2} + \|u_{xxxx}\|_{L^2} \\
\leq \|u\|_{L^2} + C \|f\|_{L^2} + C \|f\|_{L^2} + C[\|f\|_{L^2} + \|u_t\|_{L^2} + (1 + \|u(\cdot, t)\|_{L^2}) \|u_x\|_{L^2}] \\
+ C(1 + \|u(\cdot, t)\|_{L^2} + \|f\|_{L^2} + \|u_{t2}(L^2)\| + \|u(t)^2\|_{L^2} + \|u_{t2}^2(L^2)\|) \\
\leq C(\|u\|_{L^2} + \|f\|_{L^2} + \|u_t\|_{L^2}) + C(1 + \|u(\cdot, t)\|_{L^2} + \|f\|_{L^2} + \|u_{t2}(L^2)\| + \|u_{t2}^2(L^2)\| + \|u_{t2}^3(H^2)\|) \\
+ \sup_{t \leq \tau \leq t + T} (1 + \|u(\cdot, \tau)\|_{L^2}) \|u_x\|_{L^2} + \|u_{t2}(L^2)\| + \|u_t\|_{L^2(L^2)} + \|f\|_{L^2(H^2)} \\
\leq C(\|u\|_{L^2(L^2)} + \|f\|_{L^2(L^2)} + \|u_{t2}(L^2)\| + \|u_t\|_{L^2(L^2)} + \|f\|_{L^2(H^2)}) \\
+ \sup_{t \leq \tau \leq t + T} (1 + \|u(\cdot, \tau)\|_{L^2} + \|f\|_{L^2(L^2)})(\|u\|_{L^2(H^3)} + \|u_t\|_{L^2(H^2)} + \|f\|_{L^2(L^2)})
\]

By Theorem 1, we have

\[
\|u\|_{H^4} \leq C(1 + \|u(\cdot, t)\|_{L^2} + \|f\|_{L^2(L^2)})(\|u\|_{L^2(L^2)} + \|u_{t2}(L^2)\| + \|u_t\|_{L^2(L^2)} + \|f\|_{H^1(L^2)}) \\
\leq \gamma(\|\phi\|_{L^2(L^2)} + \|f\|_{L^2(L^2)})(\|u\|_{L^2(L^2)} + \|u_t\|_{L^2(L^2)} + \|f\|_{H^1(L^2)})
\]

Thus

\[
\|u\|_{L^2(H^3)} \leq \|u\|_{L^2(L^2)} + \|u_{x2}\|_{L^2(L^2)} + \|u_{xx2}\|_{L^2(L^2)} + \|u_{xxx2}\|_{L^2(L^2)} \\
\leq C(\|u\|_{L^2(L^2)} + \|f\|_{L^2(L^2)}) + C \sup_{t \leq \tau \leq t + T} (1 + \|u(\cdot, \tau)\|_{L^2}) \|u_t\|_{L^2(L^2)} + \|f\|_{L^2(L^2)} \\
\leq C \sup_{t \leq \tau \leq t + T} (1 + \|u(\cdot, \tau)\|_{L^2}) \|u_t\|_{L^2(L^2)} + \|f\|_{L^2(L^2)} + \|u_t\|_{L^2(L^2)} + \|f\|_{L^2(L^2)}
\]

By Theorem 1, we have

\[
\|u\|_{L^2(L^2)} \leq \gamma(\|\phi\|_{L^2(L^2)} + \|f\|_{L^2(L^2)})(\|u\|_{L^2(L^2)} + \|u_{t2}(L^2)\| + \|u_t\|_{L^2(L^2)} + \|f\|_{L^2(L^2)}) \\
\leq \gamma(\|\phi\|_{L^2(L^2)} + \|f\|_{L^2(L^2)})(\|u\|_{L^2(L^2)} + \|u_{t2}(L^2)\| + \|u_t\|_{L^2(L^2)} + \|f\|_{L^2(L^2)})
\]

By Lemma 3, (17), (19) follow easily. With the above definition, we have

\[
\|u\|_{y_{t2} + T} = \sup_{t \leq \tau \leq t + T} \|u\|_{H^4} + \|u\|_{L^2(H^3)} \\
\leq \sup_{t \leq \tau \leq t + T} (\|u\|_{H^4} + \|u\|_{L^2(H^3)}) \\
\leq \gamma(\|\phi\|_{L^2} + \|f\|_{L^2(L^2)}) \{2\|u(\cdot, s_1)\|_{L^2} + \|u(\cdot, s_1)\|_{L^2} e^{-\eta(t-s_1)}\}
\]
where and u first give a lemma. Conditions of Theorem 2, and the proof of Theorem 2 is meaningful. In order to prove Theorem 2, let us

\[ e^{-\eta(t-s)} \int_{s_1}^{t} e^{-2\eta(s-\tau)} (||f(\cdot, \tau)||^2_{L^2} + ||f_1(\cdot, \tau)||^2_{L^2}) d\tau \]^{1/2} + 2||f||_{L^2(L^2)} + ||f||_{H^1(L^2)} + ||f||_{L^2(H^1)}

Then (19) follows easily. By given conditions, we have

\[ 2\int_{s}^{t} e^{-2\eta(t-\tau)} (||f(\cdot, \tau)||^2_{L^2} + ||f_1(\cdot, \tau)||^2_{L^2}) d\tau \]^{1/2} \leq \begin{cases} 2Ce^{-\eta t} \sqrt{t-s} & \eta = \alpha \\ 2Ce^{-\eta t} \frac{1}{\sqrt{2(\eta-\alpha)}} [e^{2(\eta-\alpha)t} - e^{2(\eta-\alpha)s}]^{1/2} & \eta > \alpha \\ 2Ce^{-\eta t} \frac{1}{\sqrt{2(\alpha-\eta)}} [e^{2(\eta-\alpha)s} - e^{2(\eta-\alpha)t}]^{1/2} & \eta < \alpha \end{cases}\]

Hence (20) follows easily. 

We assume the external excitation f is a time period function with the period ω. For example, we can take \( f = \frac{1}{4} x^2 \sin \frac{2\pi}{t} \), then obviously \( f \in C(R^+, L^2(0, 1)) \cap L^2(R^+, H^1(0, 1)) \). And for \( ||f||_{L^2} = \frac{1}{12} \sin \frac{2\pi}{t} \), we can take \( \eta = \frac{1}{4} \in (0, \frac{1}{2}) \), then \( \sup_{0 \leq t \leq \omega} ||f(\cdot, t)||_{L^2(0, 1)} \leq \frac{1}{12} < \frac{1}{4} \). So there exists the f satisfying the conditions of Theorem 2, and the proof of Theorem 2 is meaningful. In order to prove Theorem 2, let us first give a lemma.

Let \( \phi \) be a \( s \)-compatible function in the phase space \( H^s(0, 1) \) and \( u(x, t) \) be the corresponding solution, \( u^* \) be a time period solution of the system. If we let \( w = u - u^* \), then

\[ w_t + w_{xxxx} + \lambda w_{xx} + w_w + (u^* w)_x = 0 \]

and

\[ w(x, 0) = \phi(x) - u^*(x, 0) \]

\[ w(0, t) = w(1, t) = w_x(0, t) = w_x(0, t) = 0 \]

\[ w_{xx}(1, t) = -kw_x(1, t) \]

where \( 0 < x < 1, t > 0, 0 < \lambda < 1, k \) are the same as the before defining. So we will consider the following initial boundary value problem.

\[ u_t + u_{xxxx} + \lambda u_{xx} + u_x + (v u)_x = 0, u(x, 0) = \phi(x) \]

\[ u(0, t) = u(1, t) = u_t(0, t) = u_t(0, t) = 0, u_{xx}(1, t) = -ku_x(1, t) \] (26)

where \( v \) is the given function, \( v \in C(R^+, L^4(0, 1)) \) \( \cap L^2(R^+, H^5(0, 1)) \) and \( v_t \in C(R^+, L^2(0, 1)) \) \( \cap L^2(R^+, H^5(0, 1)) \), \( 0 < x < 1, t > 0, 0 < \lambda < 1, k \) are the same as the before defining.

Likewise the discussion in the third section, we can show the global stability of the system (26) in the space \( H^j(0, 1), (j = 0, 4) \). As for the large-time behavior of the system (26), we have the following result.

**Lemma 5.** Let \( T > 0 \) be given. There exists constants \( \eta_1 \in \left( \frac{1}{2}, 1 \right) \) and \( \eta_2 \in \left( 0, \frac{1}{2} \right) \), such that if

\[ \lim_{t \to \infty} ||v||_L^2 \leq \eta_1 \] (27)

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Then for $\phi \in L^2(0, 1)$, the unique solution $u$ of the system (27) satisfies

$$\|u\|_{L^2} \leq e^{-\eta_2 t} \|\phi\|_{L^2}.$$  \hspace{1cm} (28)

For any $t \geq 0$.

**Proof:** The solution of the system (26) satisfies

$$\frac{d}{dt} \int_0^1 u^2 dx + 2kuv_x (1, t)^2 + 2 \int_0^1 u_{xx}^2 dx - 2\lambda \int_0^1 u_x^2 dx = 2 \int_0^1 vv_x dx.$$

Since $2 \|u_x\|_{L^2}^2 \leq \|u_{xx}\|_{L^2}^2$, then

$$\frac{d}{dx} \int_0^1 u^2 dx + 2 (1 - \|v (\cdot, t)\|_{L^2}) \|u_x\|_{L^2}^2 \leq 0 \hspace{1cm} (29)$$

By (27), there exists a constant $s > 0$, such that $\sup_{s \leq t < \infty} (1 - \|v (\cdot, t)\|_{L^2}) = \eta_2 > 0$. Then

$$\int_0^1 u^2 dx \leq e^{-2\eta_2 (t-s)} \|u (\cdot, s)\|_{L^2}^2$$

Integrating (29) with respect to $t$, we obtain

$$\int_t^{t+T} \frac{d}{dt} \int_0^1 u^2 (x, \tau) d\tau \leq 2\eta_2 \int_t^{t+T} \int_0^1 u_x^2 (x, \tau) dxd\tau \leq 0$$

Then we have

$$\int_t^{t+T} \int_0^1 u_x^2 (x, \tau) dxd\tau \leq \frac{1}{\eta_2} \int_0^1 u^2 dx, \quad \forall t \geq s$$

then $\|u\|_{L^2} \leq e^{-\eta_2 t} \|\phi\|_{L^2}$ follows easily. \#

**Proof of Theorem 2:** Firstly, we prove the existence of time periodic solution with the period $\omega$ satisfying the boundary conditions. If external excitation $f$ satisfies (14), choose $\phi \in H^4(0, 1)$ satisfying the compatibility condition. Let $u(x, t)$ be the solution of (1)-(4). By Lemma 4, the set $\{\|u (\cdot, t)\|_{H^4(0,1)}\}_{t=0}^{\infty}$ is uniformly bounded. Let $t_m$ be a sequence with $m \to \infty$, as $t_m \to \infty$ such that $u(1, t_m)$ converges to a function $\psi$ strongly in $L^2$ and weakly in $H^4$ as $m \to \infty$, $\psi \in H^4(0, 1)$. If we take $\psi$ as an initial data in the system (1)-(4), then the corresponding solution $u^* (x, t)$ of the system (1)-(4) is a time period function, and its period is the same as the $f$’s.

Let $u(x, t + \omega) - u(x, t) = v(x, t)$. Because of the periodicity of $f$, $v(x, t)$ satisfies the following linear problem.

$$v_t + v_{xxxx} + \lambda v_{xx} + (bv)_x = 0, v(x, 0) = \phi^* (x)$$

$$v(0, t) = v(1, t) = v_x (0, t) = v_{xx} (0, t) = 0, v_{xx} (1, t) = -kv_{xx} (1, t)$$

(30)

where $b$ is a variable coefficient,

$$b = \frac{1}{2} (u(x, t + \omega) + u(x, t)), \phi^* (x) = u(x, \omega) - u(x, 0),$$

$0 < x < 1, t > 0, 0 < \lambda < 1, k$ are the same as the before defining. Then

$$\frac{d}{dt} \int_0^1 v^2 dx + 2kv_{xx} (1, t)^2 + 2 \int_0^1 v_{xx}^2 dx - 2\lambda \int_0^1 v_x^2 dx = 2 \int_0^1 bv_{xx} dx.$$

Then we have

$$\frac{d}{dx} \int_0^1 v^2 dx + 2 \|v_{xx}\|_{L^2}^2 - 2\lambda \|v_x\|_{L^2}^2 \leq 2 \|b\|_{L^2} \|v_x\|_{L^2}^2$$

and

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\[
\frac{d}{dx} \int_0^1 v^2 dx + 2 (1 - \|b\|_{L^2}) \|v\|_{L^2}^2 \leq 0.
\]

By Gronwall inequality, we have
\[
\|v(\cdot, t)\|_{L^2} \leq \|v(\cdot, \tau)\|_{L^2} \exp \int_\tau^t 2(1 - \|b(\cdot, s)\|_{L^2}) ds,
\]

By Lemma 3, thus
\[
\lim_{t \to \infty} \|b\|_{L^2} \leq \lim_{t \to \infty} \|u\|_{L^2} \leq 2 \lim_{t \to \infty} \|f\|_{L^2} < 2\eta < 1.
\]

Hence we can choose \(\tau > 0\) which is large enough, such that
\[
2\gamma = 2 \left(1 - \sup_{t \geq \tau} \|b(\cdot, \tau)\|_{L^2}\right) > 0,
\]

then
\[
\|v(\cdot, t)\|_{L^2} \leq \|v(\cdot, \tau)\|_{L^2} e^{-\gamma (t-\tau)}, \quad \forall t \geq \tau.
\]

In particular, \(\|u(\cdot, t_m + \omega) - u(\cdot, t_m)\|_{L^2} \leq \|v(\cdot, \tau)\|_{L^2} e^{-\gamma (t_m - \tau)}\), \(\forall t_m > \tau\). Since \(u(\cdot, t_m)\) converges to \(u^*(\cdot, 0)\) strongly in \(L^2\) and \(u(\cdot, t_m + \omega)\) converges to \(u^*(\cdot, \omega)\) strongly in \(L^2\) as \(m \to \infty\). And because
\[
\|u^*(\cdot, \omega) - u^*(\cdot, 0)\|_{L^2} \leq \|u^*(\cdot, \omega) - u(\cdot, t_m + \omega)\|_{L^2} + \|u(\cdot, t_m + \omega) - u(\cdot, t_m)\|_{L^2}
\]

thus \(u^*(x, \omega) = u^*(x, 0), \forall x \in (0, 1)\), a.e., then \(u^*(x, t)\) is a time periodic function with the period \(\omega\).

Secondly, we prove the uniqueness of time periodic solution with the period \(\omega\) satisfying the boundary conditions. Let both \(u_1\) and \(u_2\) be the time period solutions of the system (1)-(4) and \(v = u_1 - u_2\). Then \(v\) satisfies linear problem (30), and where \(b = \frac{1}{2} (u_1 + u_2)\).

By Lemma 1, we have
\[
\lim_{t \to \infty} \|b(\cdot, t)\|_{L^2} \leq \frac{1}{2} \left(\lim_{t \to \infty} \|u_1(\cdot, t)\|_{L^2} + \lim_{t \to \infty} \|u_2(\cdot, t)\|_{L^2}\right) \leq 2 \lim_{t \to \infty} \|f(\cdot, t)\|_{L^2} < 1
\]

with the above proof, \(v\) decays to zero exponentially in the space \(L^2(0, 1)\). So \(u_1(x, t) = u_2(x, t), \forall x \in (0, 1)\), \(t \geq 0\), a.e.

For given \(\phi \in L^2(0, 1)\), let \(u(x, t)\) be the corresponding solution. Then \(w(x, t) = u(x, t) - u^*(x, t)\).

Solves the system (26) with \(v(x, t) = u^*(x, t)\) and \(\phi^* J(x) = \phi(x) - u^*(x, 0)\). By Lemma 3, we can choose \(\eta\) small enough such that \(\lim_{t \to \infty} \|u^*\|_{L^2} \leq \eta_1\), where \(\eta_1\) is as given in Lemma 5. Then if Lemma 5 holds, there exists \(\tau > 0, \delta > 0\) such that
\[
\|w(\cdot, t)\|_{L^2} \leq \|\phi^*(x)\|_{L^2} e^{-\delta t}, \forall t \geq 0
\]

which yields Theorem 2. #

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References


