Optimal Control of Nonlinear Strength Burgers Equation under the Neumann Boundary Condition

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Abstract: In this paper, we study the optimal control problem about a new kind of nonlinear strength Burgers equation under the Neumann boundary condition. According to Galerkin method and suitable performance index \( J(y, u) \), we give the optimal control and prove the existence of solutions of this equation in a short interval time as well as the existence of optimal solutions.

Keywords: optimal control; optimal solution; Burgers equation; nonlinear strength Burgers equation; Neumann boundary condition

1 Introduction

In this paper we study the nonlinear strength Burgers equation:

\[
y_t - \gamma y_{xx} + y^n y_x = f, \quad f \in L^2(V^*),
\]

where \( V = H^1_0(0, 1), V^* \) and \( V \) are dual spaces. It is Burgers equation as \( n = 1 \). It is the generalized Burgers equation as \( n \in N \). \( \gamma y_{xx} \) is a dissipation item (\( \gamma \) is dissipation coefficient), \( y^n y_x \) is a nonlinear strength convection item. Usually, the convection item is \( yy_x \). If a little Reynolds number is considered, we often deal convection item \( yy_x \) with \( y\infty y_x \), in which \( y\infty \) is perturbation speed. This means we can use a linear convection item \( y\infty y_x \) instead of a nonlinear convection item \( yy_x \). Predicated on this concept, we change \( yy_x \) into \( y^n y_x \). \( y^n y_x \) is a nonlinear strength convection item. This equation is of practicality, which accounts for such natural phenomena as laser wave, water wave, etc. The development of the optimal control of Burgers equation is as follows. Krtistic and Miroslav (see [1]) got some dynamics properties under the Dirichlet boundary condition and the global stability about Burgers question with boundary control. Hinze and Volkwein (see [2]) dealt Burgers equation with instant control.

In this paper we research the optimal control of the nonlinear strength Burgers equation under the Neumann boundary condition and the influence on nonlinear strength Burgers equation by nonlinear strength. The new question is different from those in [1]-[5].

Now we give the initial condition and boundary condition of (1):

\[
y(0) = \phi, \quad \phi \in H,
\]

\[
y_x(t, 0) = g_0, y_x(t, 1) = g_1, t \in [0, T],
\]

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where $T > 0, H = L^2((0, 1), \gamma > 0, y(t), f(t)$ are functions $y(t, \cdot), f(t, \cdot)$ when $t$ is fixed.

In this paper, firstly, we use Galerkin method to prove the existence of the solutions of (1)-(3). Secondly, we research the optimal control question. We choose the suitable performance index $J(y, m)$:

$$J(y, m) = \frac{1}{2} \|Cy - z\|_S^2 + \sigma \|m\|_{L^2(\omega)}^2,$$

where the operator $C \in L (W(V), S), S$ is a real Hilbert Space, $C$ is a continuance operator, $z \in S$ is a desired state, $\sigma > 0$ is a fixed constant. Optimal control question is: $\text{min} J(y, m)$. We give the optimal control of (1)-(3) and prove the existence of the optimal solutions.

2 Existence of solutions under Neumann boundary condition

Let $b(\cdot , \cdot , \cdot) = \frac{1}{n + 2} \int_0^1 (\alpha^n \beta)^{j}\mu + \alpha^n \beta^{j}\mu dx$, $\alpha, \beta, \mu \in H^1 (0, 1)$.

Definition 2.1 Let $W(V) = \{ y : y \in L^2 (V) , y_t \in L^2 (V^*) \}, y \in W(V), y$ is called a weak solution of (1.1)-(1.3) if it satisfies the equation:

$$\frac{d}{dt} \langle y(t), \varphi \rangle_H + \gamma \langle y(t), \varphi \rangle_V + b(y^n(t), y(t), \varphi) = \langle f(t), \varphi \rangle_{V^*} + g_1(t) \varphi(1) - g_0(t) \varphi(0), \forall \varphi \in V, t \in [0, T].$$

Theorem 2.1 Weak solutions of (1)-(3) exist in the interval $[0, T_0]$, in which $T_0$ is a constant ($T_0 = T_0 (\phi, f, g_0, g_1, \gamma) > 0$) and $g_0, g_1 \in L^2 (0, T), \phi \in H, f \in L^2 (V^*)$.

Proof: Firstly, we construct approximate solutions.

We apply Galerkin method to the proof. Since $V$ is a separable Hilbert space there exists an orthonormal basis $\{\psi_i\}_{i \in N}$ of $V$. Let $m \in N$ and introduce $V_m = \text{span}\{\psi_1, \cdots, \psi_m\} \subset V$. An approximate weak solution of (1)-(3) is defined as follows:

$$y_m(t) = y_m(x, t) = \sum_{i = 1}^{m} g_{im}(t) \psi_i(x),$$

where $g_{im}(t)$ ($i = 1, \cdots, m$) satisfy

$$\langle (y)_t, \psi_i \rangle - \gamma \langle (y)_x, \psi_i \rangle + \langle y^n(y), \psi_i \rangle = \langle f, \psi_i \rangle. \quad (4)$$

Because

$$y(0) = \phi, y_0(0) = \sum_{i = 1}^{m} C_{im}(t) \psi_i(x) = \phi$$

in which coefficient $C_{im}(i = 1, 2, \cdots, m)$ are constants. We have

$$y_m(0) \xrightarrow{strong} y_m(0) \text{ in } V \text{ as } m \to \infty.$$

The initial condition is $y_m(0) = \phi$ in (4). Then, we get a system of ordinary differential equations about $g_{im}(i = 1, \cdots, m)$. These equations satisfy $g_{im}(0) = C_{im}$. Because these bases are linearly independence, coefficient matrix about $g_{im}$ is invertible and local solution exists. Then the solutions $\{g_{im}(t)\}_{1 \leq i \leq m}$ about (4) exist in $[0, t_m)$. We construct a priori estimate, then we can continue the solutions from $[0, t_m)$ to $[0, T_0]$.

Secondly, we construct priori estimates. We conclude from (1) that:

$$\frac{d}{dt} \langle y_m(t), \psi_j \rangle_H + \gamma \langle y_m(t), \psi_j \rangle_V + b(y^n_m(t), y_m(t), \psi_j) = \langle f(t), \psi_j \rangle_{V^*} + g_1(t) \psi_j(1) - g_0(t) \psi_j(0), j = 1, \cdots, m, t \in [0, T],$$

$$y_m(0) = \phi_m,$$
where $\phi_m$ is the orthogonal-projection in $H$ of $\phi$ onto the space $V_m$. Hence, the estimate $\|\phi_m\|_H \leq \|\phi\|_H$ holds. Equation (5) forms a nonlinear differential system for the functions $g_{1m}, \ldots, g_{nm}$,

$$
\sum_{i=1}^{m} \langle \psi_i, \psi_j \rangle_H g_{im}(t) + \gamma \sum_{i=1}^{m} \langle \psi_i', \psi_j' \rangle_H g_{im}(t) + \sum_{i_1 \ldots i_n, l=1}^{m} \int_{0}^{t} \psi_{i_1} \psi_{i_2} \ldots \psi_{i_n} \psi_j d x g_{i_1} g_{i_2} \ldots g_{i_n} g_l =
$$

$$
\langle f(t), \psi_j \rangle_{V^*,V} + \gamma g_1(t) \psi_j(1) - \gamma g_0(t) \psi_j(0).
$$

We can write the differential equation as the form:

$$
g_{im}'(t) + \gamma \sum_{j=1}^{m} \alpha_{ij} g_{jm}(t) + \sum_{i_1 \ldots i_n, k=1}^{m} \alpha_{i_1 \ldots i_n, k} g_{i_1} g_{i_2} \ldots g_{i_n} g_{km}(t)
$$

$$
= \sum_{j=1}^{m} \beta_{ij} \left( \langle f(t), \psi_j \rangle_{V^*,V} + \gamma g_1(t) \psi_j(1) - \gamma g_0(t) \psi_j(0) \right),
$$

(6)

where $\alpha_{ij}, \alpha_{i_1 \ldots i_n, k}, \beta_{jk} \in R$, initial condition is:

$$
g_{im}(0) = \text{the } i\text{-th component of } \phi_m.
$$

The nonlinear differential system with a Lipschitz-nonlinearity (6) has a maximal solution defined in some interval around $t = 0$. Now we have proved that $y_m(t)$ exists for some interval around $t = 0$. From the theory of ordinary differential equations we can infer that in this case the solution exists as long as it is finite. Therefore, we are going to find estimates on the size of $y_m(t)$. We multiply (6) by $g_{jm}(t)$ and add these equations for $j = 1, \ldots, m$, we can derive that:

$$
\frac{1}{2} \frac{d}{dt} \|y_m(t)\|_H^2 + \gamma \|y_m(t)\|_V^2 + b(y_m^n(t), y_m(t), y_m(t)) \leq
$$

$$
\gamma \|y_m(t)\|_H^2 + \|f(t)\|_{V^*} \|y_m(t)\|_V + \gamma (|g_0(t)| + |g_1(t)|) \|y_m(t)\|_V,
$$

(7)

and

$$
b(y_m^n(t), y_m(t), y_m(t)) \leq \gamma \|y_m(t)\|_H^2 + \|f(t)\|_{V^*} \|y_m(t)\|_V + \gamma (|g_0(t)| + |g_1(t)|) \|y_m(t)\|_V.
$$

(8)

From (7) (8), we can get that

$$
\frac{1}{2} \frac{d}{dt} \|y_m(t)\|_H^2 + \gamma \|y_m(t)\|_V^2 \leq 2 \gamma \|y_m(t)\|_H^2 + 2 \|f(t)\|_{V^*} \|y_m(t)\|_V
$$

$$
+ 2 \gamma (|g_0(t)| + |g_1(t)|) \|y_m(t)\|_V.
$$

(9)

By applying the Young inequality we can derive that

$$
\|f(t)\|_{V^*} \|y_m(t)\|_V \leq \frac{1}{\gamma} \|f(t)\|_{V^*}^2 + \gamma \|y_m(t)\|_V^2,
$$

$$
\gamma (|g_0(t)| + |g_1(t)|) \|y_m(t)\|_V \leq \gamma (|g_0(t)| + |g_1(t)|)^2 + \gamma \|y_m(t)\|_V^2.
$$

So we can get:

$$
\frac{1}{2} \frac{d}{dt} \|y_m(t)\|_H^2 + \gamma \|y_m(t)\|_V^2 \leq \frac{2}{\gamma} \|f(t)\|_{V^*}^2 + \gamma \|y_m(t)\|_V^2 + 2 \gamma \|y_m(t)\|_H^2
$$

$$
+ 2 \gamma (|g_0(t)| + |g_1(t)|)^2 + \gamma \|y_m(t)\|_V^2,
$$

$$
\frac{1}{2} \frac{d}{dt} \|y_m(t)\|_H^2 \leq \frac{2}{\gamma} \|f(t)\|_{V^*}^2 + 2 \gamma \|y_m(t)\|_H^2 + 2 \gamma \left( |g_0(t)|^2 + |g_1(t)|^2 \right),
$$

$$
\frac{d}{dt} \|y_m(t)\|_V^2 \leq \frac{4}{\gamma} \|f(t)\|_{V^*}^2 + 4 \gamma \|y_m(t)\|_H^2 + 4 \gamma \left( |g_0(t)|^2 + |g_1(t)|^2 \right).
$$

Define $\omega_m(t) = 1 + \|y_m(t)\|_H^2 \geq 1$, then we have

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Finally, for \( \frac{d\omega_m}{dt} \leq \frac{4}{\gamma} \|f(t)\|^2_{Y^*} + 4\gamma \|y_m(t)\|^2_H + 4\gamma \left( |g_0(t)|^2 + |g_1(t)|^2 \right) \leq \frac{4}{\gamma} \|f(t)\|^2_{Y^*} + 4\gamma \left( |g_0(t)|^2 + |g_1(t)|^2 \right) \right] \omega_m^2(t), \]
\[
\int_{\omega_m(0)}^{\omega_m(t)} \frac{d\omega_m(t)}{\omega_m^2(t)} \leq \int_0^t \frac{4}{\gamma} \|f(s)\|^2_{Y^*} + 4\gamma \left( |g_0(s)|^2 + |g_1(s)|^2 \right) ds,
\]
\[
\frac{1}{\omega_m(0)} - \frac{1}{\omega_m(t)} \leq \frac{4}{\gamma} \|f(t)\|^2_{L^2(0,T;V^*)} + 4\gamma t + 4\gamma \left( |g_0(t)|^2_{L^2(0,T)} + |g_1(t)|^2_{L^2(0,T)} \right). \tag{10}
\]
By the dominated convergence theorem: \( \exists T_* \in [0,T], \)
\[
\|f\|_{L^2(0,T;V^*)} \leq \sqrt{\frac{\gamma}{6}} \left( 1 + \|\phi\|^2_H \right), \|g_i\|_{L^2(0,T)} \leq \frac{1}{\sqrt{24\gamma}} \left( 1 + \|\phi\|^2_H \right), (i = 0, 1).
\]
Using (10), we have
\[
\frac{-1}{\omega_m(t)} \leq \frac{4}{\gamma} \|f(t)\|^2_{L^2(0,T;V^*)} + 4\gamma t + 4\gamma \left( |g_0(t)|^2_{L^2(0,T)} + |g_1(t)|^2_{L^2(0,T)} \right) - \frac{1}{\omega_m(0)},
\]
\[
\frac{1}{\omega_m(t)} \geq \frac{1}{\|\phi\|^2_H} - 4\gamma t - \frac{1}{\|\phi\|^2_H},
\]
\[
\frac{1}{\omega_m(t)} \geq \frac{\left( 1 + \|\phi\|^2_H \right)^2 - 4\gamma t \left( 1 + \|\phi\|^2_H \right) \left( 1 + \|\phi\|^2_H \right) - \left( 1 + \|\phi\|^2_H \right)^2 - \left( 1 + \|\phi\|^2_H \right)}{\left( 1 + \|\phi\|^2_H \right)^2 - 4\gamma t \left( 1 + \|\phi\|^2_H \right) \left( 1 + \|\phi\|^2_H \right) - \left( 1 + \|\phi\|^2_H \right)^2 - \left( 1 + \|\phi\|^2_H \right)}, \forall t \in (0,T_*].
\]
Finally, for \( \forall t \in (0,T_*], \) we derive that:
\[
\omega_m(t) \leq \frac{\left( 1 + \|\phi\|^2_H \right)^2 - 4\gamma t \left( 1 + \|\phi\|^2_H \right) \left( 1 + \|\phi\|^2_H \right) - \left( 1 + \|\phi\|^2_H \right)^2 - \left( 1 + \|\phi\|^2_H \right)}{\left( 1 + \|\phi\|^2_H \right)^2 - 4\gamma t \left( 1 + \|\phi\|^2_H \right) \left( 1 + \|\phi\|^2_H \right) - \left( 1 + \|\phi\|^2_H \right)^2 - \left( 1 + \|\phi\|^2_H \right)},
\]
where
\[
\left( 1 + \|\phi\|^2_H \right)^2 - 4\gamma t \left( 1 + \|\phi\|^2_H \right) \left( 1 + \|\phi\|^2_H \right) - \left( 1 + \|\phi\|^2_H \right)^2 - \left( 1 + \|\phi\|^2_H \right) > 0.
\]
\[
\|y_m(t)\|^2_H \leq \omega_m(t) \leq \frac{\left( 1 + \|\phi\|^2_H \right)^2 - 4\gamma t \left( 1 + \|\phi\|^2_H \right) \left( 1 + \|\phi\|^2_H \right) - \left( 1 + \|\phi\|^2_H \right)^2 - \left( 1 + \|\phi\|^2_H \right)}{\left( 1 + \|\phi\|^2_H \right)^2 - 4\gamma t \left( 1 + \|\phi\|^2_H \right) \left( 1 + \|\phi\|^2_H \right) - \left( 1 + \|\phi\|^2_H \right)^2 - \left( 1 + \|\phi\|^2_H \right)} = d_1.
\]
Let
\[
T_0 = \min \left( T_*, \frac{\left( 1 + \|\phi\|^2_H \right)^2 - \left( 1 + \|\phi\|^2_H \right)}{4\gamma \left( 1 + \|\phi\|^2_H \right) \left( 1 + \|\phi\|^2_H \right)^2} \right),
\]
for \( \forall m > 0, t \in [0, T_0], \) we can get \( y_m \in L^\infty([0,T_0];H) \) and \( \|y_m\|^2_{L^\infty([0,T_0],L^2(0,1))} \leq d_1. \) By integrating (8) in the interval \([0,T_0]\) for \( t \) and using (11), it follows that:
\[
\|y_m(T_0)\|^2_H \leq \|\phi\|^2_H + 4\gamma \int_0^{T_0} \|y_m(s)\|^2_{V^*} ds + \frac{4}{\gamma} \|f\|^2_{L^2(V^*)} + 4\gamma \left( |g_0(t)|^2_{L^2(0,T)} + |g_1(t)|^2_{L^2(0,T)} \right).
\]
Hence, \( y_m \) is uniformly bounded and independent of \( d_1 \) and \( m \).

Now we prove a bound for \( \|y_m\|_{L^2([0,T_0];V^*)} \):

\[
\|y_m\|_{L^2([0,T_0];V^*)}^2 = \int_0^{T_0} \left( \sup_{\|\varphi\|_V = 1} \langle f(\varphi) - \gamma \varphi, \varphi \rangle + 32 \|y_m(t)\|_{V^*}^2 \right) dt
\]

\[
\leq \int_0^{T_0} \left( \|f(t)\|_{V^*} + \gamma \|g_0(t)\| + \|g_1(t)\| + \|y_m(t)\|_{H^1} + 2 \|y_m\|_{L^2([0,T_0];V)}^2 \right) dt
\]

\[
\leq 2 \|f\|_{L^2([0,T_0];V^*)}^2 + 8 \gamma \|g_0\|_{V^*}^2 + 8 \gamma \|y_m\|_{L^2([0,T_0];V)}^2 + 16 \gamma \|y_m\|_{L^\infty([0,T_0];H)}^2 + 32 \|y_m\|_{L^2([0,T_0];V)}^2 = d_2,
\]

\( \forall m > 0, t \in [0,T_0], \|y_m\|_{L^2([0,T_0];V^*)} \leq d_2 \).

Therefore \( y_m \) is bounded in \( W([0,T_0];V^*) \). Then we have a subsequence \( \{y_m\}_{m \in \mathbb{N}} \) and an element \( y \in W([0,T_0];V^*) \), we get \( y_m \to y \) as \( m \to \infty \).

So we get:

\[
\frac{d}{dt} \langle y_m(t), \psi_j \rangle_{H^1} + \gamma \langle y_m(t), \psi_j \rangle_{V^*} - \gamma \langle y_m(t), \psi_j \rangle_{H^1} + b(y^n(t), y(t), \psi_j) = \langle f(t), \varphi_j \rangle_{V^*} + \gamma \langle g(t), \psi_j \rangle (1) - \gamma \langle g_0(t), \psi_j \rangle (0).
\]

This conclusion is true for any \( \psi \in V \).

Finally, we prove the limit about the existence of the approximate solutions.

Because \( \|y_m(t)\|^2_{L^\infty([0,T_0];L^2(0,1))} \leq d_1 \) and \( d_1 \) is independent of \( m \), \( y_m \) belongs to the bounded set in \( L^\infty(0,T_0, L^2(0,1)) \) as \( m \to \infty \),

\[
y_m \xrightarrow{weak^*} y, \quad in \quad L^\infty(0,T_0, L^2(0,1)). \quad (11)
\]

Because \( \|y_m\|_{L^2([0,T_0];V^*)} \leq d_2 \) and \( d_2 \) is independent of \( m \), \( (y_m)_t \) belongs to the bounded set in \( L^\infty(0,T_0, H^1_0(0,1)) \) as \( m \to \infty \). So there is a subsequence \( \{(y_m)_t\} \), which satisfies:

\[
(y_m)_t \xrightarrow{weak^*} y_t, \quad in \quad L^\infty([0,T_0], H^1_0(0,1)). \quad (12)
\]

So \( y_m \) belongs to a bounded set in \( H^1_0(0,1) \). According to the Rellich theorem, we can get \( y_m \to y \), in \( L^2(0,1) \).

We also know there is a subsequence which is convergent to \( y \).

We study the nonlinear item \( h(y) = y^n y_x \). First, because \( \|y_m\|^2_{L^\infty([0,T_0];L^2(0,1))} \leq d_1 \) and the constant \( d_1 \) is independent of \( m \), we have:

\[
(y_m)_x \xrightarrow{weak^*} y_x, \quad in \quad L^\infty([0,T], H^1_0(0,1)). \quad (13)
\]

With (2)(4)(11)-(13) we can get:

\[
h(y_m) \xrightarrow{weak^*} h(y), \quad in \quad L^\infty([0,T], H^1_0(0,1)). \quad (14)
\]

With (2)(4)(13)-(14) we can get:

\[
(y_m)_t \xrightarrow{weak^*} (y_t, \psi_j), \quad in \quad L^\infty(0,T_0),
\]

\[
(h(y_m), \psi_j) \xrightarrow{weak^*} (h(y), \psi_j), \quad in \quad L^\infty(0,T_0).
\]

Let \( m > j \), then

\[
((y_m)_t, \psi_j) - \gamma ((y_m)_xx \psi_j) + (y_m^n (y_m)_x \psi_j) = (f, \psi_j).
\]
When \( m \to \infty \) for all \( j \) we have:

\[
(y_t, \psi_j) - \gamma (y_{xx}, \psi_j) + (y^n y_x, \psi_j) = (f, \psi_j) .
\]

Because \( \{ \psi_j \} \) is dense in \( H^1_0 (0, 1) \), then we have:

\[
(y_t, v) - \gamma (y_{xx}, v) + (y^n y_x, v) = (f, v), \forall v \in H^1_0 (0, 1) .
\]

Now we verify that \( y \) satisfies the initial condition. Because \( y_m \xrightarrow{weak} y \) in \( L^\infty ([0, T]; L^2 (0, 1)) \), we can get:

\[
\int_0^T (y_m, v) dt \to \int_0^T (y, v) dt, \forall v \in L^1 ([0, T]; L^2 (0, 1)) ,
\]

\[
\int_0^T (y_m, v) dt \to \int_0^T (y, v) dt, \forall v \in L^1 ([0, T]; L^2 (0, 1)) .
\]

Let

\[
v (t, x) = \theta (t) \omega (x), \omega \in L^2 (0, 1), \theta \in C^1 (0, T),
\]

which satisfies \( \theta (0) = 1 \) and \( \theta (T) = 0 \). If we let \( v = \theta \omega, v = \theta \omega \), we can get:

\[
\lim_{m \to \infty} (y_m (0), \omega) = (y (0), \omega), \forall \omega \in L^2 (0, 1) .
\]

This means \( y_m (0) \xrightarrow{weak} y (0) \) in \( L^2 (0, 1) \). Thus \( y(0) = \phi \). Hence \( y \) satisfies the initial condition.

**Remark:** Theorem 2.1 means there are two constants \( C_0 > 0, C_1 > 0 \), which satisfy

\[
\| y \|_{W^1([0,T],V)} = \sqrt{\| y_1 \|^2_{L^2(V)} + \| y_t \|^2_{L^2(V^*)}} = C_0 \left( \| g_0 \|^2_{L^2(0,T)} + \| g_1 \|^2_{L^2(0,T)} \right) + C_1 ,
\]

\( C_1 \) depends on \( f \) and \( \phi \).

### 3 Existence of optimal control solutions

Let \( Q = (0, T) \times (0, 1), \omega \subseteq Q \) is an open set with positive measure, let \( V = H^1_0 (0, 1) \),

\[
H = L^2 (0, 1), f \in L^2 (H), \phi \in H, B \in L (L^2 (\omega), L^2 (H)) , Bq = \{ q \in Q_0, q \in Q/\omega \} .
\]

We study the nonlinear strength Burgers equation with the control item. Let \( m \in L^2 (\omega) \) and \( y \in W (V) \) are given by the weak solution of:

\[
y_t - \gamma y_{xx} + y^n y_x = f + Bm, \quad f \in L^2 (V^*),
\]

\[
y (0) = \phi, \quad \phi \in H,
\]

\[
y_x (t, 0) = g_0, y_x (t, 1) = g_1, \quad t \in [0, T].
\]

We know (1)-(3) exist weak solutions by theorem 2.1. Given an operator \( C \in L (W (V), S) \), in which \( S \) is a real Hilbert space, \( C \) is continuous. We choose a suitable performance index to these equations:

\[
J (y, m) = \frac{1}{2} \| Cy - z \|^2_S + \frac{\sigma}{2} \| m \|^2_{L^2(\omega)} .
\]

\( z \) is a desired state and \( \sigma > 0 \) is fixed. Control problem about (17)-(19) is:

\[
\min J (y, m) ,
\]

where \( (y, m) \) satisfies (17)-(19).
We introduce the operator and define \( e = (e_1, e_2) \):

\[
e(y, m) = \begin{bmatrix}
(-\Delta)^{-1} (y_t + y^n y_x - \gamma y_{xx} - f - Bm) \\
y'(, 0) - \phi \\
y_x'(t, 0) - g_0, y_x'(t, 1) - g_1
\end{bmatrix},
\]

where \( \Delta \) is an operator from \( H_0^1(\Omega) \) to \( H^{-1}(\Omega) \). Then we can get:

\[
\min_J (y, m), \text{ s.t. } e(y, m) = 0. \quad (22)
\]

**Theorem 3.1** There are optimal control solutions about (22).

**Proof:** Let \( (y, m) \in X \) and \( X = W(V) \times L^2(\varepsilon) \) satisfy the equation \( e(y, m) = 0 \), we have \( J(y, m) \geq \frac{\sigma}{2} \|m\|_{L^2(\omega)}^2 \). From Theorem 2.1 we can conclude that \( \|y\|_{W(V)} \to \infty \) yields \( \|m\|_{L^2(\omega)} \to \infty \). Hence, for \( \|(y, m)\|_X \to \infty \), we can get:

\[
J(y, m) \to \infty.
\]

Because \( C \) is continuous and the inner product \( \langle \cdot, \cdot \rangle_S \) is weakly continuous, we let \( \{(y^k, m^k)\} \) \( k \in N \) is a sequence in \( X \) converging weakly to \((y^*, m^*)\). It follows that:

\[
\lim_{n \to \infty} \inf J\left(y^k, m^k\right) \geq \frac{1}{2} \|Cy^*\|^2 - \langle Cy^*, Z \rangle_S + \frac{1}{2} \|z\|^2 + \frac{\sigma}{2} \|y^*\|_{L^2}^2 = J(y^*, m^*).
\]

Then \( J \) is weak lowered semi-continuous.

As \( J(y, m) \geq 0 \) for all \( (y, m) \in X \) holds, there is a \( \zeta \geq 0 \):

\[
\zeta = \inf \{J(y, m) : (y, m) \in X, e(y, m) = 0\}.
\]

This implies the existence of a minimizing sequence \( \{(y^k, m^k)\}_{k \in N} \) in \( X \) such that

\[
\zeta = \lim_{n \to \infty} J(y^k, m^k), \quad e(y^k, m^k) = 0, \forall n \in N.
\]

Since \( J \) is unbounded, there exists an element \((y^*, m^*) \in X \) with

\[
y^k \to y^*, k \to \infty, y \in W(V), \quad (23)
\]

\[
m^k \to m^*, k \to \infty, \quad m \in L^2(W). \quad (24)
\]

We can infer from (24) that:

\[
\lim \int_0^T \left( y^k(t) - y^*(t), \varphi(t) \right)_{V^*, V} dt = 0, \quad \forall \varphi \in L^2(V),
\]

\[
y^k \to y^*, \quad \text{in } L^2(V). \quad (25)
\]

Since \( W(V) \to L^2(L^\infty) \) holds, we have

\[
y^k \to y^* \quad \text{in } L^2(L^\infty). \quad (26)
\]

As the sequence \( \{y^k\}_{k \in N} \) converges weakly, \( \|y^k\|_{W(V)} \) is bounded. \( \|y^k\|_{c(H)} \) and \( \|y^k\|_{L^2(V^*)} \) are also bounded. Thus, it follows by Holder’s inequality that

\[
\left| \int_0^T \int_0^1 \left( \left( y^k \right)^n y^k_x - (y^*)^n y^*_x \right) \varphi dx dt \right|
\]

\[
= \int_0^T \int_0^1 \left[ \left( \left( y^k \right)^n y^k_x \right)^2 - \left( y^* \right)^n y^*_x \right] \varphi dx dt
\]

\[
= \int_0^T \int_0^1 \left( \left( y^k \right)^n y^k_x - (y^*)^n y^*_x \right) \varphi dx dt
\]

\[
= \int_0^T \int_0^1 \left( \left( y^k \right)^n (y^k - y^*), (y^k - y^*) \right) \varphi dx dt
\]

\[
= \int_0^T \int_0^1 \left( \left( y^k \right)^n \left( y^k - y^* \right)_x + (y^k - y^*), \left( y^k \right)^{n-1} + (y^k)^{n-2} y^* \ldots + (y^*)^{n-1} \right) \varphi dx dt
\]

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\[ \left\| \left( y^k \right)^n \right\|_{L^2(L^\infty)} \left\| y^k - y^* \right\|_{L^2(V)} \left\| \varphi \right\|_{L^2(V)} + \left\| y^k - y^* \right\|_{L^2(L^\infty)} \left( \left( y^k \right)^{n-1} + (y^*)^{n-1} \right)_{L^2(L^\infty)} \leq \left\| y^* \right\|_{L^2(V)} \left\| \varphi \right\|_{L^2(V)} \to 0, \quad \forall \varphi \in L^2(V). \quad (27) \]

By (24), we get:
\[ \int_0^T \int_0^1 (m^n - m^*) \varphi dx dt \to 0, n \to \infty. \quad (28) \]

From \( y^* \in W(V) \), we can derive that \( y^* (0) \in H \). Since \( y^n \to y^* \), we can infer \( y^n (0) \to y^* (0) \) for \( \forall \psi \in H \), thus:
\[ \left( y^n (0) - y^* (0), \psi \right)_H \to 0, (n \to \infty). \]

By (25),(26),(27),(28), we can conclude \( e(y^*, m^*) = 0 \) in \( L^2(V^*) \times H \).

**References**


