ACADIMIC World Academic Union

ISSN 1749-3889(print),1749-3897(online)
International Journal of Nonlinear Science
Vol.2(2006) No.2,pp.67-76

On the Well-posedness Problem for the Generalized Degasperis-Procesi Equation

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(Received 17 July 2006, accepted 30 August 2006)

Abstract: In this paper, we study the local well-posedness of the Cauchy problem for the generalized Degasperis-Proces equation. By applying some Sobolev's inequalities and related knowledge of PDE and using Kato's theory, we prove that there is a unique local solution of this problem which continuously depends on the initial value.

Key words: generalized D-P equation ; Kato's theory;local well posedness;infinite generator

1 Introduction

Recently, Degasperis and Procesi ([1]) studied the following family of third order dispersive PDE conservation laws,

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x , \tag{1.1}$$

where α,c_0,c_1,c_2 and c_3 are real constants. They found that there are not less than four equations, which satisfy completely integrability condition within this family :the KdV eqution, the Camassa-Holm equation, the Dullin-Gottwald-Holm equation and the Degasperis-Procesi equation.

With $\alpha = c_2 = c_3 = 0$ in Eq.(1.1), it becomes the Well-known Korteweg-deVeris equation. The KdV equation is completely integrable and its solitary waves are solitions([2, 3]). The Cauchy problem of the KdV equation has been the subject of a number of studies, and a satisfactory local or global existence theory is now in hand([4]). It is shown that as long as $u_0 \in H^s(R)\,, s \geq 1$, the KdV equation is globally well-posed. It is observed that the KdV equation does not have breaking wave.

For $c_1 = -\frac{3}{2}c_3/\alpha^2, c_2 = c_3/2$, Eq.(1.1)becomes the Camassa-Holm equation. It has a bi-Hamiltionian structure and is completely integrable([5]). The Cauchy problem of the Camassa-Holm equation has been studied extensively. It has been shown that the Camassa-Holm equation is locally well-posedness ([6, 7]) with the initial data $u_0 \in H^s(R)\,, s > \frac{3}{2}$. More interestingly, it has global strong solutions and also blow-up solutions in finite time ([8, 9, 26])with a different class of initial profiles in the Sobolev spaces H^S, $s > \frac{3}{2}$. On the other hand, it has global weak solutions in $H^1(R)$. In [10],Ding and Tian researched solutions of the dissipative Camassa-Holm equation on Total space. The advantage of the Camassa-Holm equation in comparison with the KdV equation is clear: the C-H equation has peaked solutions, but no shock waves.

With $c_1 = -2c_3/\alpha^2, c_2 = c_3$ in Eq.(1.1), we find the Degasperis-Procesi equation of the form

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx} \quad t > 0\,,\ x \in R. \tag{1.2}$$

Degasperis,Holm and Hone([11])proved the exact integrability of Eq.(1.2)by constructing a Lax pair. They also showed that Eq.(1.2)has bi-Hamiltonian structure and an infinite sequence of conserved quantities,and

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IJNS.2006.10.15/034

admits exact peakon solutions which are analogous to the Camassa-Holm equation . The Degasperis-Procesi equation can be regard as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the C-H shallow water equation. Dullin, Gottwald and Holm([12]) showed that the D-P equation can be obtained from the shallow water equation by an appropriate Kodama transformation. Vakhnenko and Parkes ([13]) investigated traveling wave solutions of Eq.(1.2). Holm and Staley studied stability of solutions and peakons numerically to Eq.(1.2).

After the $D-P$ equation (1.2) was derived , many papers were devoted to its study. For example, Yin proved local well-posedness to Eq.(1.2) with initial data $u_0 \in H^s(R)$, $(s > \frac{3}{2})$ on the line ([14]) and on the circle ([24]) and derived the precise blow-up scenario and a blow-up result. The global existence of strong solutions and global weak solutions to Eq.(1.2) are also investigated in([15, 16]). Recently, Lenells ([17]) classified all weak traveling wave solutions. Matsuno ([18]) studied multisoliton solutions and their peakon limit.

In this paper, we are interested in the Cauchy problem for a generalized case of equation (1.2). Thus we replace the term $4uu_x$ in Eq.(1.2) with $4u^m u_x (m > 0)([25, 27])$. So we derive the initial value problem of the generalized Degasperis-Procesi equation(GDP),or call it the modified Degasperis-Procesi (mDP) as:

$$\begin{cases} u_t - u_{txx} + 4u^m u_x = 3u_x u_{xx} + uu_{xxx}, t > 0, x \in R, \\ u(0,x) = u_0(x), x \in R. \end{cases} \tag{1.3}$$

Now we study GDP equation (1.3) and give the motivation of such modification. The GDP equation is difference from DP equation. The nonlinear convection term uu_x is changed into the new nonlinear convection term $u^m u_x$. Considering the problem of small Reynolds number, it is usually delayed with nonlinear convection term uu_x into $u_\infty u_x$, where u_∞ is the speed unperturbed, or $u\frac{\partial u}{\partial x} \approx u_\infty \frac{\partial u}{\partial x}$. For this reason, in general case, if we deal with the term by the method, the nonlinear term is changed into a linear term and new structures will be caused, and the peakon solution will not be solution of the new modified equation, and large error could arise. The nonlinear essence of nonlinear convection term dose not appear. Hence we use the way of improving the order of u which is the coefficient of gradient term $\frac{\partial u}{\partial x}$, that is , we change $u(\frac{\partial u}{\partial x})$ into $u^m(\frac{\partial u}{\partial x})$,$m \in N$, where m is representative of the strength of the nonlinearity and work out its solution.Then we get the new equation(1.3), and call it GDP equation, or the modified D-P (mDP) equation. We consider the influence on the solution of the equation with the increasing of the order of nonlinear terms and many new traveling wave solutions, new solitary wave solutions will be found out.At the same time,its interaction with new nonlinear convection generates new structures. Under this structure, Eq.(1.3) will create many new nonlinear phenomenon, such as compacton solutions with compact support, solitons with cusps, or peaks. In this paper, we focus our attention on the local well-posedness of GDP equation.

We shall use the following notations without further comment. Let $\|\cdot\|_X$ be the norm of the Banach space X; $B(X,Y)$ denotes the space of all bounded linear operator from X to $Y(B(X)$ if $X=Y)$; Let $D(A)$ be the domain of the operator A , $\partial = \partial_x = \frac{\partial}{\partial_x}; \Lambda^s = (1-\partial_x^2)^{s/2}$, $s \in R$; $H^s = H^s(R)$ with norm $\|f\|_{H^s} = \|f\|_s = (\int_R (1+|\xi|^2)^s \left|\hat{f}(\xi)\right|^2 d\xi)^{1/2}$ and $\langle\cdot,\cdot\rangle_s$ for its inner product; $[A,B]$ denotes the commutator of the linear operators A and B.

2 Kato's theory

For convenience, we state here Kato's theorem in a form suitable for our purpose. Consider the abstract quasi-linear evolution equation([22],[23]):

$$\frac{dv}{dt} + A(v)v = f(v), t \geq 0, v(0) = v_0. \tag{2.1}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X, and let Q : $Y \to X$ be a topological isomorphism. Let $L(Y,X)$ denote the space of all bounded linear operators from Y to X.If $X=Y$, we denote this space by $L(X)$. We make the following assumptions, where μ_1, μ_2, μ_3, and μ_4 are constants depending only on $\max\{\|y\|_Y, \|z\|_Y\}$:

(i) $A(y) \in L(Y,X)$ for $y \in X$ with

$$\|(A(y)-A(z))w\|_X \le \mu_1 \|y-z\|_X \|w\|_Y, \quad y,z,w\in Y$$

and $A(y)\in G(X,1,\beta)$ (i.e. $A(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(ii) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y)\in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\|(B(y)-B(z))w\|_X \le \mu_2 \|y-z\|_Y \|w\|_X, \quad y,z\in Y, w\in X.$$

(iii) $f: Y\to Y$ extends to a map from X into X, is bounded on bounded sets in Y , and satisfies

$$\|f(y)-f(z)\|_Y \le \mu_3 \|y-z\|_Y, \quad y,z\in Y,$$

$$\|f(y)-f(z)\|_X \le \mu_4 \|y-z\|_X, \quad y,z\in Y.$$

Theorem 2.1 (Kato, [19]).*Assume that (i), (ii), and (iii) hold. Given $v_0\in Y$, there is a maximal $T>0$ depending only on $\|v_0\|_Y$, and a unique solution v to (2.1) such that*

$$v = v(\cdot, v_0) \in C([0,T);Y)\cap C^1([0,T);X).$$

Moreover, the map $v_0\to v(\cdot,v_0)$ is a continuous map from Y to $C([0,T);Y)\cap C^1([0,T);X)$.

Lemma 2.1 ([19]) Let γ, t be real numbers such that $-\gamma < t \le \gamma$. Then,

$$\|fg\|_t \le c\|f\|_\gamma \|g\|_t, \quad \gamma > \frac{1}{2}$$

$$\|fg\|_{\gamma+t-\frac{1}{2}} \le c\|f\|_\gamma \|g\|_t, \quad \gamma < \frac{1}{2}$$

where c is a positive constant depending on γ and t.

Lemma 2.2 ([20])Let$f\in H^s, s>\frac{3}{2}$. Then,

$$\left\|\wedge^{-\gamma}\left[\wedge^{\gamma+t+1}, Mf\right]\wedge^{-t}\right\|_{L(L^2)} \le c\|f\|_s, \quad |\gamma|, |t| \le s-1$$

where M_f is the operator of multiplication by f, and c is a constant depending only on γ and t.

Lemma 2.3 Let $f,g\in H^s$ and $s>\frac{1}{2}$, then

$$\|fg\|_s \le c\|f\|_s\|g\|_s.$$

That's because H^s is a Banach algebra for $s>\frac{1}{2}$.

Lemma 2.4 Let $s>\frac{3}{2}$, then

$$\|u_x\|_{L^\infty} \le \|u\|_s.$$

This lemma is derived directly from the Sobolev embedding theorem.

Lemma 2.5 ([21, §4.5, Theorems 5.5 and 5.8]) . Let X and Y be two Banach spaces and Y be continuously and densely embedded in X. Let $-A$ be the infinitesimal generator of the C_0-semigroup $T(t)$ on X and let S be an isomorphism from Y onto X. Then Y is $-A$-admissible if and only if $-A_1 = -SAS^{-1}$ is the infinitesimal generator of the C_0-semigroup $T_1(t) = ST(t)S^{-1}$ on X. Moreover, if Y is $-A$-admissible, then the part of $-A$ in $Y$is the infinitesimal generator of the restriction of $T(t)$ to Y.

3 Local well-posedness

With $y := u - u_{xx}$, (1.3) takes the form of a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} y_t + uy_x + 3u_x y + 4u_x(u^m - u) = 0, t > 0, x \in R, \\ y(0,x) = u_0(x) - u_{0,xx}(x), x \in R. \end{cases} \quad (3.1)$$

We also note that if $p(x) := \frac{1}{2}e^{-|x|}$, $x \in$R , then $\left(1-\partial_x^2\right)^{-1} f = p*f$ for all $f \in L^2(R)$, and $p*y = u$. Using this identity, we can rewrite (3.1) as

$$\begin{cases} u_t + uu_x = -\partial_x p * (\frac{4}{m+1}u^{m+1} - \frac{1}{2}u^2), t > 0, x \in R, \\ u(0,x) = u_0(x), x \in R, \end{cases} \quad (3.2)$$

or, equivalently, as

$$\begin{cases} u_t + uu_x = -\partial_x (1-\partial_x^2)^{-1} (\frac{4}{m+1}u^{m+1} - \frac{1}{2}u^2), t > 0, x \in R, \\ u(0,x) = u_0(x), x \in R. \end{cases} \quad (3.3)$$

Theorem 3.1 *Given $u_0 \in H^s(R)$, $s > \frac{3}{2}$, there exists a maximal value $T = T(u_0) > 0$, and a unique solution u to (1.3) (or (3.3)), such that*

$$u = u(\cdot, u_0) \in C([0,T); H^s) \cap C^1\left([0,T); L^2\right)$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$u_0 \to u(\cdot, u_0)\, H^s \to C([0,T); H^s) \cap C^1\left([0,T); L^2\right)$$

is continuous.

To prove this result, we will apply Theorem 2.1 with

$$A(u) = u\partial_x, \quad f(u) = -\partial_x(1-\partial_x^2)^{-1}(\frac{4}{m+1}u^{m+1} - \frac{1}{2}u^2),$$

$Y = H^s, X = L^2, \wedge = \left(1-\partial_X^2\right)^{\frac{1}{2}}$,and $Q = \wedge^s$. Obviously, Q is an isomorphism of H^s onto H^{s-1}. Thus, in order to derive Theorem 3.1 from Theorem 2.1, we only need to verify that $A(u)$ and $f(u)$ satisfy the conditions(i),(ii),(iii).

Lemma 3.1 The operator $A(u) = u\partial_x$, with $u \in H^s, s > \frac{3}{2}$, belongs to $G\left(L^2, 1, \beta\right)$.

Proof. Since L^2 is a Hilbert space, we have $A(u) \in G\left(L^2, 1, \beta\right)$ for some real number β if and only if the following conditions hold ([22]):

(a) $(A(u)y, y)_0 \geq -\beta \|y\|_0^2$.

(b) The range of $A + \lambda$ is all of X, for some (or all) $\lambda > \beta$.

We first prove (a). Since $u \in H^s, s > \frac{3}{2}$, u and u_x belong to L^∞. Note that $\|u_x\|_{L^\infty} \leq \|u\|_s$. Thus

$$(A(u)y, y)_0 = (u\partial_x y, y)_0 = -\frac{1}{2}(u_x y, y)_0 \leq \frac{1}{2}\|u_x\|_{L^\infty}\|y\|_0^2 \leq c\|u\|_s \|y\|_0^2.$$

Setting $\beta = c\|u\|_s$, we obtain $(A(u)y, y)_0 \geq -\beta\|y\|_0^2$.

Next, we prove (b). Because A(u) is a closed operator and satisfies (a), $(\lambda I + A)$ has closed range in L^2 for all $\lambda > \beta$. Therefore, it suffices to show that $(\lambda I + A)$ has dense range in L^2 for all $\lambda > \beta$.

Given $u \in H^s, s > \frac{3}{2}$, and $y \in L^2$, we have the generalized Leibnitz formula

$$\partial_x(uy) = u_x y + u\partial_x y \quad in\ H^{-1}.$$

Since $u_x \in L^\infty$, we have

$$\begin{aligned} D(A) &= D(u\partial_x) = \left\{y \in L^2, u\partial_x y \in L^2\right\} \\ &= \left\{z \in L^2, -\partial_x(uz) \in L^2\right\} \\ &= D((u\partial_x)^*) = D(A^*.) \end{aligned}$$

Assume that the range of $(A + \lambda I)$ is not all of L^2, then there exists $z \in L^2, z \neq 0$ such that $((\lambda I + A)\, y, z)_0 = 0$ for all $y \in D(A)$. Since $H^1 \subset D\,(A)$, $D\,(A)$ is dense in L^2. Hence it follows that $z \in D\,(A^*)$ and $\lambda z + A^* = 0$ in L^2. Since $D\,(A) = D\,(A^*)$, multiplying by z and integrating by parts, we obtain

$$0 = ((\lambda I + A^*)\, z, z)_0 = (\lambda z, z) + (z, Az) \geq (\lambda - \beta)\, \|z_0\|_0^2\,, \quad \forall \lambda > \beta.$$

and thus $z = 0$, which contradicts our assumption $z \neq 0$. This completes the proof of Lemma 3.1.

Lemma 3.2 Let $A(u) = u\partial_x$ with $u \in H^s, s > \frac{3}{2}$. Then $A\,(u) \in L\left(H^s, L^2\right)$ for all $u \in H^s$. Moreover,

$$\|(A\,(u) - A\,(z))\, w\|_0 \leq \mu_1 \|u - z\|_0 \|w\|_s\,, \quad u, z, w \in H^s$$

Proof. Let $u, z, w \in H^s, s > \frac{2}{3}$. Then we have

$$\|(A(u) - A(z))w\|_0 \leq c\, \|u - z\|_0 \|\partial_x w\|_{L_\infty} \leq \mu_1 \|u - z\|_0 \|w\|_s$$

Taking $z = 0$ in the above inequality, we obtain $A\,(u) \in L\left(H^s, L^2\right)$. This completes the proof of Lemma 3.2.

Lemma 3.3 We have $B\,(u) = [\wedge^s, u\partial_x] \wedge^{-s} \in L\left(L^2\right)$, for $u \in H^s$. Moreover,

$$\|(B\,(u) - B\,(z))\, w\|_0 \leq \mu_2 \|u - z\|_s \|w\|_0\,.$$

Proof. Let $u, z \in H^s, s > \frac{2}{3}$, and $w \in L^2$. Then

$$\begin{aligned} \|(B(u) - B(z))w\|_0 &= \left\|[\Lambda^s, (u - v)\partial_x]\, \Lambda^{-s} w\right\|_0 \\ &\leq \left\|[\Lambda^s, (\mathrm{u} - \mathrm{v})]\, \Lambda^{1-\mathrm{s}}\right\|_{L(L^2)} \left\|\Lambda^{-1}\partial_x w\right\|_0 \\ &\leq \mu_2 \|y - z\|_s \|w\|_0\,, \end{aligned}$$

where we applied Lemma 2.2 with $\gamma = 0, t = s - 1$. Taking $z = 0$ in the above inequality, we obtain $B\,(u) \in L\left(L^2\right)$. This completes the proof of Lemma 3.3.

Lemma 3.4 Let $f(u) = -\partial_x(1 - \partial_x^2)^{-1}(\frac{4}{m+1}u^{m+1} - \frac{1}{2}u^2)$. Then f is bounded on bounded sets inH^s, and satisfies

$$(a)\, \|f\,(y) - f\,(z)\|_s \leq \mu_3 \|y - z\|_s\,, \quad y, z \in H^s,$$

$$(b)\, \|f\,(y) - f\,(z)\|_0 \leq \mu_4 \|y - z\|_0\,, \quad y, z \in L^2.$$

Proof. Let $y, z \in H^s, s > \frac{3}{2}$, and note that H^{s-1} is a Banach algebra. Then we have

$$\begin{aligned} \|f\,(y) - f\,(z)\|_s &= \left\|-\partial_x(1 - \partial_x^2)^{-1}\left[\frac{4}{m+1}(y^{m+1} - z^{m+1}) - \frac{1}{2}(y^2 - z^2)\right]\right\|_s \\ &\leq \frac{4}{m+1}\left\|y^{m+1} - z^{m+1}\right\|_{s-1} + \frac{1}{2}\left\|y^2 - z^2\right\|_{s-1}. \end{aligned}$$

And

$$\begin{aligned}\left\|y^{m+1}-z^{m+1}\right\|_{s-1} &= \left\|(y-z)(y^m+y^{m-1}z+\cdots+z^m)\right\|_{s-1}\\ &\le \|y-z\|_s\left\|y^m+y^{m-1}z+\cdots+z^m\right\|_s\\ &\le \|y-z\|_s\left(\|y\|_s^m+\|y\|_s^{m-1}\|z\|_s+\cdots+\|z\|_s^m\right)\\ &\triangleq K_1\|y-z\|_s, \quad (K_1<\infty)\end{aligned}$$

$$\left\|y^2-z^2\right\|_{s-1}\le\|y-z\|_s\|y+z\|_s\le(\|y\|_s+\|z\|_s)\|y-z\|_s\triangleq K_2\|y-z\|_s, \quad (K_2<\infty).$$

This proves (a).

Taking $z=0$ in the above inequality, we obtain that f is bounded on bounded sets in H^s.

Next, we will prove (b). Let $y,z\in H^s, s>\frac{3}{2}$, and note that H^{s-1} is a Banach algebra.

Then we have

$$\begin{aligned}\|f(y)-f(z)\|_0 &= \left\|-\partial_x(1-\partial_x^2)^{-1}\left[\frac{4}{m+1}(y^{m+1}-z^{m+1})-\frac{1}{2}(y^2-z^2)\right]\right\|_0\\ &\le \frac{4}{m+1}\left\|y^{m+1}-z^{m+1}\right\|_0+\frac{1}{2}\left\|y^2-z^2\right\|_0.\end{aligned}$$

And

$$\begin{aligned}\left\|y^{m+1}-z^{m+1}\right\|_0 &= \left\|(y-z)(y^m+y^{m-1}z+\cdots+z^m)\right\|_0\\ &\le \|y-z\|_0\left\|y^m+y^{m-1}z+\cdots+z^m\right\|_s\\ &\le \|y-z\|_0\left(\|y\|_s^m+\|y\|_s^{m-1}\|z\|_s+\cdots+\|z\|_s^m\right)\\ &\triangleq K_3\|y-z\|_0, \quad (K_3<\infty);\end{aligned}$$

$$\begin{aligned}\left\|y^2-z^2\right\|_0 &\le \|y-z\|_0\|y+z\|_s\\ &\le (\|y\|_s+\|z\|_s)\|y-z\|_0\\ &\triangleq K_4\|y-z\|_0, \quad (K_4<\infty).\end{aligned}$$

which proves (b). This completes the proof of Lemma 3.4.

Proof of Theorem 3.1 The result follows by combining Theorem 2.1 and Lemmas 3.1-3.4.

Lemma 3.5 The operator $A(u)=u\partial_x$, with $u\in H^s, s>\frac{3}{2}$, belongs to $G(H^{s-1},1,\beta)$.

Proof. Since H^{s-1} is a Hilbert space, $A(u)$ belongs to $G(H^{s-1},1,\beta)$ for some real number β if and only if the following conditions hold ([22]):

$(a)(\mathrm{A(u)}y,y)_{s-1}\ge-\beta\|y\|_{s-1}^2$.

$(b)-A(u)$is the infinitesimal generator of a C_0-semigroup on H^{s-1}, for some (or all) $\lambda>\beta$.

We first prove (a). Since $u\in H^s, s>\frac{3}{2}$, it follows that u and u_x belong to L^∞ and $\|u_x\|_{L^\infty}\le\|u\|_s$. Note that

$$\Lambda^{s-1}(u\partial_x y)=\left[\Lambda^{s-1},u\right]\partial_x y+u\Lambda^{s-1}(\partial_x y)=\left[\Lambda^{s-1},u\right]\partial_x y+u\partial_x\Lambda^{s-1}y.$$

Thus

$$\begin{aligned}(A(u)y,y)_{s-1} &= (\Lambda^{s-1}(u\partial_x y),\Lambda^{s-1}y)_0\\ &= (\left[\Lambda^{s-1},u\right]\partial_x y,\Lambda^{s-1}y)_0-\frac{1}{2}(u_x\Lambda^{s-1}y,\Lambda^{s-1}y)_0\\ &\le \left\|\left[\Lambda^{s-1},u\right]\Lambda^{2-s}\right\|_{L(L_2)}\left\|\Lambda^{s-1}y\right\|_0^2+\|u_x\|_{L^\infty}\left\|\Lambda^{s-1}y\right\|_0^2\\ &\le c\|u\|_s\|y\|_{s-1}^2.\end{aligned}$$

by Lemma 2.2 with $\gamma=0, t=s-2$. Setting $\beta=c\|u\|_s$, we obtain $(\mathrm{A(u)}y,y)_{s-1}\ge-\beta\|y\|_{s-1}^2$, as claimed.

Next, we prove (b). Let $S = \Lambda^{s-1}$, and note that S is an isomorphism of H^{s-1} onto L^2 and that H^{s-1} is continuously and densely embedded in L^2 since $s > \frac{3}{2}$. Define

$$A_1(u) := SA(u)S^{-1} = \Lambda^{s-1}A(u)\Lambda^{1-s} \quad , \quad B_1(u) := A_1(u) - A(u).$$

Let $y \in L^2$ and $u \in H^s, s > \frac{3}{2}$. Then we have

$$\begin{aligned} \|B_1(u)y\|_0 &= \left\| \left[\Lambda^{s-1}, u\partial_x\right] \Lambda^{1-s}y \right\|_0 \\ &\leq \left\| \left[\Lambda^{s-1}, u\right] \Lambda^{2-s} \right\|_{L(L_2)} \left\| \Lambda^{-1}\partial_x y \right\|_0 \\ &\leq c\|u\|_s \|y\|_0 \end{aligned}$$

by Lemma 2.2 with $\gamma = 0, t = s - 2$. Hence $B_1(u) \in L(L_2)$.

Note that $A_1(t) = A(t) + B_1(t)$ and $A(u) \in G\left(L^2, 1, \beta\right)$ in Lemma 3.1. By a perturbation theorem for semigroups (cf. [20, §5.2, Theorem 2.3]) we obtain $A_1(u) \in G\left(L^2, 1, \beta\right)$. Applying Lemma 2.5 with $Y = H^{s-1}$, $X = L^2$, and $S = \Lambda^{s-1}$, we conclude that H^{s-1} is A-admissible. Hence$-A(u)$ is the infinitesimal generator of a C_0-semigroup on H^{s-1}. This completes the proof of Lemma 3.5.

Theorem 3.2 *The maximal T in Theorem 3.1 may be chosen independent of s in the following sense. If*

$$u = u(\cdot, u_0) \in C([0,T); H^s) \cap C^1\left([0,T); H^{s-1}\right)$$

is a solution to (1.3) (or (3.3)), and if $u_0 \in H^{s'}$ for some $s' \neq s, s' > \frac{3}{2}$, then

$$u \in C\left([0,T); H^{s'}\right) \cap C^1\left([0,T); H^{s'-1}\right),$$

with the same value of T. In particular, if $u_0 \in H^\infty = \bigcap_{s \geq 0} H^s$, then $u \in C([0,T); H^\infty)$.

Proof. If $s' < s$, the result follows the uniqueness of the solution guaranteed by Theorem 3.1, so it suffices to consider the case $s' > s$. To this end, we return to (3.1). Setting $y(t) = \wedge^2 u(t) = u - u_{xx}$, we have

$$\frac{dy}{dt} + A(t)y + B(t)y = f(t), \quad y(0) = \wedge^2 u(0), \tag{3.4}$$

where $A(t)y = 3\partial_x(uy)$,$B(t)y = -2uy_x$ and $f(t) = 4u_x(u - u^m)$

Because $u \in C([0,T); H^s)$ and $u_0 \in H^{s'}$, we have

$$y \in C\left([0,T); H^{s-2}\right)$$

and

$$y(0) = \left(1 - \partial_x^2\right) u(0) \in C\left([0,T); H^{s'}\right).$$

We will show that $y \in C\left([0,T); H^{s'-2}\right)$, which implies $u \in C\left([0,T); H^{s'}\right)$ since $\left(1 - \partial_x^2\right)$ is an isomorphism from $H^{s'}$ to $H^{s'-2}$. This will complete the proof of Theorem 3.2.

Since $u \in C([0,T); H^s)$, $u_x \in H^{s-1}$, and H^{s-1} is a Banach algebra, we have

$$B(t) \in L\left(H^{s-1}\right), f(t) \in C\left([0,T); H^{s-1}\right).$$

Following the arguments in Lemmas 3.1-3.3 in [20], we first prove that the family $A(t)$ has a unique evolution operator $\{U(t,\tau)\}$ associated with the spaces $X = H^h$ and $Y = H^k$, where $-s \leq h \leq s - 2$; $1 - s \leq k \leq s - 1$, and $k \geq h + 1$. To this end, as in the proof of Lemma 3.1 in [20], we need to verify the following three conditions:

(i)$A(t) \in G\left(H^h, 1, \beta\right), \forall y \in H^s$

(ii)$\wedge^h \partial_x \left[\wedge^{k-h}, u\right] \wedge^{-k}$ is uniformly bounded on L^2.

(iii)$A(t) \in L\left(H^k, H^h\right)$ is strongly continuous in t.

Let us first show (i). Since H^h is a Hilbert space, we have $A(t) \in G(H^h, 1, \beta)$ for some real number β if and only if the following conditions hold ([22]):

(a) $(A(t)y, y)_h \geq -\beta \|y\|_h^2$

(b) $-A(t)$is the infinitesimal generator of a C_0-semigroup on H^h, for some (or all) $\lambda > \beta$.

To prove (a), take $y \in H^h$ and note that

$$\begin{aligned} \Lambda^h \partial_x(3uy) &= 3\Lambda^h \partial_x(-\left[\Lambda^{-h}, u\right] \Lambda^h y + \Lambda^{-h}(u\Lambda^h y)) \\ &= -3\Lambda^h \partial_x \left[\Lambda^{-h}, u\right] \Lambda^h y + 3\partial_x(u\Lambda^h y). \end{aligned}$$

Thus

$$\begin{aligned} (A(t)y, y)_h &= (-3\Lambda^h \partial_x \left[\Lambda^{-h}, u\right] \Lambda^h y + 3\partial_x(u\Lambda^h y), \Lambda^h y)_0 \\ &= 3(\wedge^{h+1}\left[\wedge^{-h}, u\right] \wedge^h y, \partial_x \wedge^{h-1} y)_0 + \frac{3}{2}\left(u_x \wedge^h y, \wedge^h y\right)_0 \\ &\leq 3\left\|\wedge^{h+1}\left[\Lambda^{-\mathrm{h}}, \mathrm{u}\right]\right\|_{L(L^2)} \left\|\wedge^h y\right\|_0^2 + \frac{3}{2}\|u_x\|_{L^\infty} \left\|\wedge^h y\right\|_0^2 \\ &\leq c\|u\|_s \|\mathrm{y}\|_h^2, \end{aligned}$$

where we have used Lemma 2.2 with $\gamma = -(h+1)$ and $t = 0$. Setting $\beta = c\|u\|_s$, we obtain $(A(t)y, y)_h \geq -\beta\|y\|_h^2$, as claimed.

Next, we prove (b). Let $S = \wedge^{s-1-h}$, and note that S is an isomorphism of H^{s-1} onto H^h and that H^{s-1} is continuously and densely embedded in H^h as $-s \leq h \leq s-2$. Define

$$A_1(t) := SA(t)S^{-1} = \wedge^{s-1-h}A(t)\wedge^{h+1-s},$$

$$B_1(t) := A_1(t) - A(t) = [S, A(t)]S^{-1}.$$

Let $y \in H^h$ and $u \in H^s, s > \frac{3}{2}$. Then

$$\begin{aligned} \|B_1(t)y\|_h &= \left\|\Lambda^h \partial_x \left[\Lambda^{s-1-h}, 3u\right] \Lambda^{h+1-s} y\right\|_0 \\ &\leq \left\|\wedge^h \partial_x \left[\wedge^{s-1-h}, 3u\right] \wedge^{1-s}\right\|_{L(L^2)} \left\|\wedge^h y\right\|_0 \\ &\leq c\|u\|_s \|y\|_h \end{aligned}$$

on applying Lemma 2.2 with $\gamma = -(h+1), t = s-1$. Therefore, we have $B_1(t) \in L(H^h)$. Since $A(t)y = \partial_x(3uy) = 3u_x y + 3u\partial_x y$ and $u_x \in L(H^{s-1})$, by applying Lemma 3.5 and a perturbation theorem for semigroups, we see that H^{s-1} is A(t)-admissible. Further, applying Lemma 2.5 with $Y = H^{s-1}$,$X = H^h$ and $S = \wedge^{s-1-h}$, we obtain that $-A_1(t)$ is the infinitesimal generator of a C_0-semigroup on H^h. Since $A_1(t) = A(t) + B_1(t)$ and $B_1(t) \in L(H^h)$, by a perturbation theorem for semigroups it follows that $-A(t)$ is the infinitesimal generator of a C_0-semigroup on H^h. This proves (b).

Next, we verify (ii). For $y \in L^2$ we have

$$\left\|\wedge^h \partial_x \left[\wedge^{k-h}, u\right] \wedge^{-k} y\right\|_0 \leq c\|u\|_s \|y\|_0,$$

by Lemma 2.2 with $\gamma = -(h+1), t = k$. This proves (ii).

Finally, we verify (iii). Take $y \in H^k$. Then

$$\begin{aligned} \|(A(t+\tau) - A(t))y\|_h &= \|3\partial_x(u(t+\tau) - u(t))y\|_h \\ &\leq 3\|((u(t+\tau) - u(t))y)\|_{h+1} \\ &\leq c\|u(t+\tau) - u(t)\|_{s-1}\|y\|_{h+1} \\ &\leq c\|u(t+\tau) - u(t)\|_s \|y\|_k \end{aligned}$$

by Lemma 2.2 with $\gamma = s-1, t = h+1$. The continuity of u now yields (iii).

Thus, the above conditions (i)-(iii) imply the existence and uniqueness of an evolution operator $U(t,\tau)$ for the family $A(t)$. In particular, for $-s \le \gamma \le s-1$, $U(t,\tau)$ maps H^γ into itself.

Next, take $Y = H^{s-2}$,$X = H^{s-3}$, and note that

$$y \in C\left([0,T); H^{s-1}\right) \cap C^1\left([0,T); H^{s-2}\right).$$

Using the properties of the evolution operator $U(t,\tau)$, we obtain

$$\frac{d}{d\tau}\left(U(t,\tau)\, y(\tau)\right) = U(t,\tau)\left(-B(\tau)\, y(\tau) + f(\tau)\right).$$

An integration over $\tau \in [0,t]$ gives

$$y(t) = U(t,0)\, y(0) + \int_0^t U(t,\tau)\left(-B(\tau)\, y(\tau) + f(\tau)\right) d\tau. \tag{3.5}$$

If $s < s' \le s+1$, then $f(t) \in C\left([0,T); H^{s-1}\right) \cap C^1\left([0,T); H^{s'-2}\right)$, $B(t) = -2u\partial_x \in L\left(H^{s'-2}\right)$ is strongly continuous on $[0,t)$, and $H^{s-1}H^{s'-2} \subset H^{s'-2}$ since $s-1 > \frac{1}{2}$. Since $-s < s-2 < s'-2 \le s-1$, the family $\{U(t,\tau)\}$ is a strongly continuous map from the space $H^{s'-2}$ into itself. Noting that $y(0) \in H^{s'-2}$, and regarding (3.5) as an integral equation of Volterra type that can be solved for y by successive approximation, we then obtain the assertion of Theorem 3.2 for the case $s < s' \le s+1$.

In the case $s' > s+1$, the result follows by a repeated application of the above argument. This completes the proof of Theorem 3.2.

Acknowledgements

This research is supported by Outstanding Personnel Program in Six Fields of Jiangsu Province(NO.6-A-029) and the Teaching and Research Award Program for Outstanding Young Teachers in higher Education Institutions of MOE,P.R.C.(NO.2002-383)

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