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Lipschitz Equivalence Between Two Sierpinski Gasketse

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Abstract: Recently, a lot of work has been devoted to the study of the Lipschitz equivalence between self-similar sets. Many results have been well-known under the condition that the similarity ratios equal each other of the real line. In this paper, we generalize some results to a more general setting. We mainly study the Lipschitz equivalence between two Sierpinski gaskets.

Keywords: Lipschitz equivalence; bi-Lipschitz transformation; Sierpinski gasket; invariant set; graph-directed set

1 Introduction

Chaos research is one of the focuses in nonlinear science. The chaotic attractors almost have fractal construction such as butterfly-shaped attractor. They can be applied to many natural and engineering systems. On the other hand, with the development of the research of the chaos control, the research of chaotic synchronization also attracts many researchers' attention[1-3]. A lot of work has been devoted to the study of an attractor with fractal construction[4-6]. In some situations, we concern about the relationship between two attractors. In this paper, we mainly study the Lipschitz equivalence between two fractal attractors.

Two compact metric spaces (A, d_A) and (B, d_B) are Lipschitz equivalence, and then we write $A \simeq B$. If there is a bijection f from A to B and a constant $c > 0$ such that, for all $x_1, x_2 \in A$, we have

$$c^{-1} d_A(x_1, x_2) \leqslant d_B(f(x_1), f(x_2)) \leqslant c d_A(x_1, x_2). \tag{1.1}$$

The corollary 2.4 in [7] revealed a fundamental property of Hausdorff dimension: Hausdorff dimension is invariant under bi-Lipschitz transformations. Thus if two sets have different dimensions there cannot exist a bi-Lipschitz mapping from one onto the other. Meanwhile, it is well-known that there are invariant sets P, Q with $\dim_H P = \dim_H Q$ such that they are not Lipschitz equivalent. For example, Wen and Xi[8] constructed self-similar arcs with the same Hausdorff dimension but not Lipschitz equivalent.

Recently, many results have been established in the study of Lipschitz equivalence between two dust-like self-similar sets[9] and between dust-like self-conformal sets[10]. But the dust-like condition is too strong for contractions. Indeed, very little is known about the Lipschitz equivalence between self-similar sets. It can be seen from an open question in [11]: Problem 11.16 of [11] poses the question whether the two self-similar sets M and M' defined by

$$M = (M/5) \cup (M/5 + 2/5) \cup (M/5 + 4/5) \quad M' = (M'/5) \cup (M'/5 + 3/5) \cup (M'/5 + 4/5), \tag{1.2}$$

are Lipschitz equivalent. The authors of [9] gave a positive answer. In this paper, we generalize the result in [12] to a more general setting.

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2 Notations

Let $\{f_i\}_{i=1}^N$ be a family of contractive similitudes in $\mathbb{R}^n$. Then there is a unique non-empty compact set E called the invariant set or self-similar set of system $\{f_i\}_{i=1}^N$ such that $E = \bigcup_{i=1}^{N} f_i(E)$[13]. If $f_i(E) \cap f_j(E) = \emptyset$ for all $i \neq j$, the self-similar E is called dust-like.

Let $\{f_i\}_{i=1}^N$ and $\{g_i\}_{i=1}^N$ be two families of contractive similitudes of the real line whose invariant set are E and F. Suppose E and F are dust-like, and the contraction ratios of f_i and g_i are equal for all $1 \leqslant i \leqslant N$. Then it is easy to show that E and F are Lipschitz equivalent. This result can be generalized to graph-directed sets, which play an important role in our study.

We recall the definition of graph-directed sets[14]. Let $G = (V, \Gamma)$ be a directed graph with vertex $V = \{1, \cdots N\}$ and directed-edge set Γ. We assume that for any $1 \leqslant i \leqslant N$, there is at least one edge starting from vertex i. For any edge $e \in \Gamma$, there is a corresponding similitude $T_e : \mathbb{R}^n \to \mathbb{R}^n$ with similitude ratio $\rho_e \in (0, 1)$. That is, $|T_e(x) - T_e(y)| = \rho_e|x - y|, \forall x, y \in \mathbb{R}^n$. The graph G labeled with the similitudes T_e will be denoted by G^*. We call G the base graph of G^*. The set of edges from i to j is denoted by $\Gamma_{i,j}$. The graph-directed sets on G^* are the unique non-empty compact sets $\{E_i\}_{i=1}^N$ satisfying

$$E_i = \bigcup_{j=1}^{N} \bigcup_{e \in \Gamma_{i,j}} T_e(E_j), 1 \leqslant i \leqslant N.$$

We call $\{E_i\}_{i=1}^N$ dust-like, if the above union is a disjoint union for each i.

Theorem 2.1.[12] Let $\{E_i\}_{i=1}^N$ and $\{F_i\}_{i=1}^N$ be the graph-directed sets on G^* and H^* respectively. If
(1) The base graphs coincide, i.e., $G = H$;
(2) For each edge e of G, the similitude S_e and T_e have the same ratio ρ_e;
(3) $\{E_i\}_{i=1}^N$ and $\{F_i\}_{i=1}^N$ are dust-like,
then, $E_i \simeq F_i$ holds for $1 \leqslant i \leqslant N$.

With the result in graph-directed sets, the authors of [12] proved that the two self-similar sets M and M' defined by (1.2) are Lipschitz equivalent.

3 Lipschitz equivalence of two Sierpinski gaskets

The two Sierpinski gaskets we study here are generated by two families of contractive similitudes. Notice that all the similarity ratios in (1.2) equal $\frac{1}{5}$ of the real line. Now we begin to consider the case that the similarity ratios equal $\frac{1}{5}$ in $\mathbb{R}^2$.

Definition 3.1. We establish an orthogonal coordinate system as follows. Take the origin to be a vertex of D. Then $D = [0, 1] \times [0, 1]$ and E can be regarded as the attractor of the iterated function system $\{f_1, f_2, \cdots, f_9\}$ with the stronger separation condition, where $f_i(x) = \frac{x}{5} + b_i$, $i = 1, 2, \cdots, 9$, and $b_1 = (0, 0)$, $b_2 = (\frac{2}{5}, 0)$, $b_3 = (\frac{4}{5}, 0)$, $b_4 = (0, \frac{2}{5})$, $b_5 = (\frac{2}{5}, \frac{2}{5})$, $b_6 = (\frac{4}{5}, \frac{2}{5})$, $b_7 = (0, \frac{4}{5})$, $b_8 = (\frac{2}{5}, \frac{4}{5})$, $b_9 = (\frac{4}{5}, \frac{4}{5})$. We have the Sierpinski gasket $E = \bigcup_{i=1}^{9} f_i(E)$, which is shown in Figure 1.

The other set F can be regarded as the attractor of the iterated function system $\{g_1, g_2, \cdots, g_9\}$, where $g_i(x) = \frac{x}{5} + c_i$, $i = 1, 2, \cdots, 9$, and $c_1 = (0, 0)$, $c_2 = (\frac{3}{5}, 0)$, $c_3 = (\frac{4}{5}, 0)$, $c_4 = (0, \frac{3}{5})$, $c_5 = (\frac{3}{5}, \frac{3}{5})$, $c_6 = (\frac{4}{5}, \frac{3}{5})$, $c_7 = (0, \frac{4}{5})$, $c_8 = (\frac{3}{5}, \frac{4}{5})$, $c_9 = (\frac{4}{5}, \frac{4}{5})$. We have the Sierpinski gasket $F = \bigcup_{i=1}^{9} g_i(F)$, which is shown in Figure 2.

Theorem 3.2. For two Sierpinski gaskets E and F defined by Definition 3.1, we have $E \simeq F$.

Proof. We first consider the set F, let the following sets
$F_1 = F$,
$F_2 = F \cup (F + (1, 0))$,
$F_3 = F \cup (F + (0, 1))$,
$F_4 = F \cup (F + (1, 0)) \cup (F + (2, 0))$,
$F_5 = F \cup (F + (0, 1)) \cup (F + (0, 2))$,

$$\begin{aligned}
F_6 &= F \cup (F+(1,0)) \cup (F+(0,1)) \cup (F+(1,1)),\\
F_7 &= F \cup (F+(1,0)) \cup (F+(2,0)) \cup (F+(0,1)) \cup (F+(1,1)) \cup (F+(2,1)),\\
F_8 &= F \cup (F+(1,0)) \cup (F+(0,1)) \cup (F+(1,1)) \cup (F+(0,2)) \cup (F+(1,2)),\\
F_9 &= F \cup (F+(1,0)) \cup (F+(2,0)) \cup (F+(0,1)) \cup (F+(1,1))\\
&\qquad \cup (F+(2,1)) \cup (F+(0,2)) \cup (F+(1,2)) \cup (F+(2,2)).
\end{aligned}$$

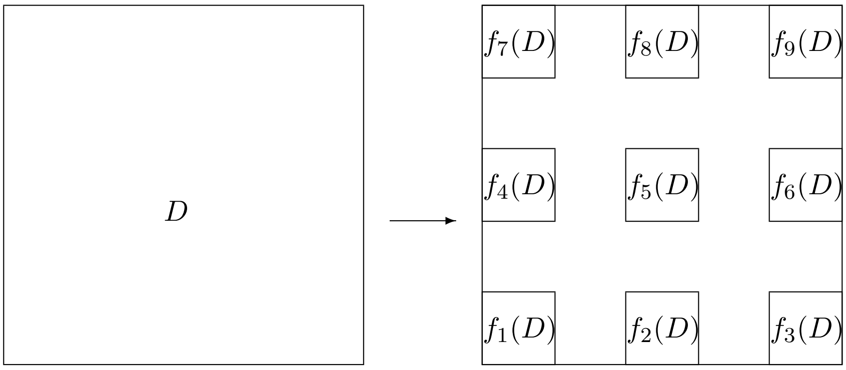

Figure 1: 1st generation of E

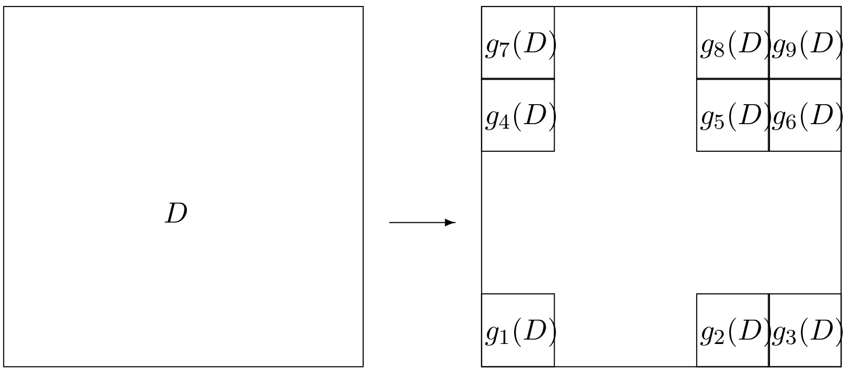

Figure 2: 1st generation of F

From $g_i(x) = \frac{x}{5} + c_i$, $i = 1, 2, \cdots, 9$, we can obtain that

$$\begin{aligned}
F_1 &= \tfrac{F}{5} \cup (\tfrac{F}{5} + (\tfrac{3}{5},0)) \cup (\tfrac{F}{5} + (\tfrac{4}{5},0)) \cup (\tfrac{F}{5} + (0,\tfrac{3}{5})) \cup (\tfrac{F}{5} + (\tfrac{3}{5},\tfrac{3}{5}))\\
&\qquad \cup (\tfrac{F}{5} + (\tfrac{4}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5} + (0,\tfrac{4}{5})) \cup (\tfrac{F}{5} + (\tfrac{3}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5} + (\tfrac{4}{5},\tfrac{4}{5}))\\
&= \tfrac{F_1}{5} \cup (\tfrac{F_2}{5} + (\tfrac{3}{5},0)) \cup (\tfrac{F_3}{5} + (0,\tfrac{3}{5})) \cup (\tfrac{F_6}{5} + (\tfrac{3}{5},\tfrac{3}{5})),\\
F_2 &= F \cup (F+(1,0))\\
&= \tfrac{F}{5} \cup (\tfrac{F}{5} + (\tfrac{3}{5},0)) \cup (\tfrac{F}{5} + (\tfrac{4}{5},0)) \cup (\tfrac{F}{5} + (0,\tfrac{3}{5})) \cup (\tfrac{F}{5} + (\tfrac{3}{5},\tfrac{3}{5}))\\
&\qquad \cup (\tfrac{F}{5} + (\tfrac{4}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5} + (0,\tfrac{4}{5})) \cup (\tfrac{F}{5} + (\tfrac{3}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5} + (\tfrac{4}{5},\tfrac{4}{5}))\\
&\qquad \cup (\tfrac{F}{5} + (1,0)) \cup (\tfrac{F}{5} + (\tfrac{8}{5},0)) \cup (\tfrac{F}{5} + (\tfrac{9}{5},0)) \cup (\tfrac{F}{5} + (1,\tfrac{3}{5})) \cup (\tfrac{F}{5} + (\tfrac{8}{5},\tfrac{3}{5}))\\
&\qquad \cup (\tfrac{F}{5} + (\tfrac{9}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5} + (1,\tfrac{4}{5})) \cup (\tfrac{F}{5} + (\tfrac{8}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5} + (\tfrac{9}{5},\tfrac{4}{5}))\\
&= \tfrac{F_1}{5} \cup (\tfrac{F_4}{5} + (\tfrac{3}{5},0)) \cup (\tfrac{F_2}{5} + (\tfrac{8}{5},0)) \cup (\tfrac{F_3}{5} + (0,\tfrac{3}{5})) \cup (\tfrac{F_7}{5} + (\tfrac{3}{5},\tfrac{3}{5})) \cup (\tfrac{F_6}{5} + (\tfrac{8}{5},\tfrac{3}{5})),\\
F_3 &= F \cup (F+(0,1))\\
&= \tfrac{F_1}{5} \cup (\tfrac{F_2}{5} + (\tfrac{3}{5},0)) \cup (\tfrac{F_5}{5} + (0,\tfrac{3}{5})) \cup (\tfrac{F_8}{5} + (\tfrac{3}{5},\tfrac{3}{5})) \cup (\tfrac{F_3}{5} + (0,\tfrac{8}{5})) \cup (\tfrac{F_6}{5} + (\tfrac{3}{5},\tfrac{8}{5})),\\
F_4 &= F \cup (F+(1,0)) \cup (F+(2,0))\\
&= \tfrac{F}{5} \cup (\tfrac{F}{5} + (\tfrac{3}{5},0)) \cup (\tfrac{F}{5} + (\tfrac{4}{5},0)) \cup (\tfrac{F}{5} + (0,\tfrac{3}{5})) \cup (\tfrac{F}{5} + (\tfrac{3}{5},\tfrac{3}{5}))\\
&\qquad \cup (\tfrac{F}{5} + (\tfrac{4}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5} + (0,\tfrac{4}{5})) \cup (\tfrac{F}{5} + (\tfrac{3}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5} + (\tfrac{4}{5},\tfrac{4}{5}))\\
&\qquad \cup (\tfrac{F}{5} + (1,0)) \cup (\tfrac{F}{5} + (\tfrac{8}{5},0)) \cup (\tfrac{F}{5} + (\tfrac{9}{5},0)) \cup (\tfrac{F}{5} + (1,\tfrac{3}{5})) \cup (\tfrac{F}{5} + (\tfrac{8}{5},\tfrac{3}{5}))\\
&\qquad \cup (\tfrac{F}{5} + (\tfrac{9}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5} + (1,\tfrac{4}{5})) \cup (\tfrac{F}{5} + (\tfrac{8}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5} + (\tfrac{9}{5},\tfrac{4}{5}))\\
&\qquad \cup (\tfrac{F}{5} + (2,0)) \cup (\tfrac{F}{5} + (\tfrac{13}{5},0)) \cup (\tfrac{F}{5} + (\tfrac{14}{5},0)) \cup (\tfrac{F}{5} + (2,\tfrac{3}{5})) \cup (\tfrac{F}{5} + (\tfrac{13}{5},\tfrac{3}{5}))\\
&\qquad \cup (\tfrac{F}{5} + (\tfrac{14}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5} + (2,\tfrac{4}{5})) \cup (\tfrac{F}{5} + (\tfrac{13}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5} + (\tfrac{14}{5},\tfrac{4}{5}))\\
&= \tfrac{F_1}{5} \cup (\tfrac{F_4}{5} + (\tfrac{3}{5},0)) \cup (\tfrac{F_4}{5} + (\tfrac{8}{5},0)) \cup (\tfrac{F_2}{5} + (\tfrac{13}{5},0)) \cup (\tfrac{F_3}{5} + (0,\tfrac{3}{5}))\\
&\qquad \cup (\tfrac{F_7}{5} + (\tfrac{3}{5},\tfrac{3}{5})) \cup (\tfrac{F_7}{5} + (\tfrac{8}{5},\tfrac{3}{5})) \cup (\tfrac{F_6}{5} + (\tfrac{13}{5},\tfrac{3}{5})),
\end{aligned}$$

$$\begin{aligned}
F_5 &= F \cup (F+(0,1)) \cup (F+(0,2)) \\
&= \tfrac{F_1}{5} \cup (\tfrac{F_2}{5}+(\tfrac{3}{5},0)) \cup (\tfrac{F_5}{5}+(0,\tfrac{3}{5})) \cup (\tfrac{F_8}{5}+(\tfrac{3}{5},\tfrac{3}{5})) \cup (\tfrac{F_5}{5}+(0,\tfrac{8}{5})) \\
&\quad\ \cup(\tfrac{F_8}{5}+(\tfrac{3}{5},\tfrac{8}{5})) \cup (\tfrac{F_3}{5}+(0,\tfrac{13}{5})) \cup (\tfrac{F_6}{5}+(\tfrac{3}{5},\tfrac{13}{5})), \\
F_6 &= F \cup (F+(1,0)) \cup (F+(0,1)) \cup (F+(1,1)) \\
&= \tfrac{F}{5} \cup (\tfrac{F}{5}+(\tfrac{3}{5},0)) \cup (\tfrac{F}{5}+(\tfrac{4}{5},0)) \cup (\tfrac{F}{5}+(0,\tfrac{3}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{3}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5}+(0,\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{4}{5})) \\
&\quad \cup(\tfrac{F}{5}+(1,0)) \cup (\tfrac{F}{5}+(\tfrac{8}{5},0)) \cup (\tfrac{F}{5}+(\tfrac{9}{5},0)) \cup (\tfrac{F}{5}+(1,\tfrac{3}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{3}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5}+(1,\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{4}{5})) \\
&\quad \cup(\tfrac{F}{5}+(0,1)) \cup (\tfrac{F}{5}+(\tfrac{3}{5},1)) \cup (\tfrac{F}{5}+(\tfrac{4}{5},1)) \cup (\tfrac{F}{5}+(0,\tfrac{8}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{8}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{8}{5})) \cup (\tfrac{F}{5}+(0,\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{9}{5})) \\
&\quad \cup(\tfrac{F}{5}+(1,1)) \cup (\tfrac{F}{5}+(\tfrac{8}{5},1)) \cup (\tfrac{F}{5}+(\tfrac{9}{5},1)) \cup (\tfrac{F}{5}+(1,\tfrac{8}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{8}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{8}{5})) \cup (\tfrac{F}{5}+(1,\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{9}{5})) \\
&= \tfrac{F_1}{5} \cup (\tfrac{F_4}{5}+(\tfrac{3}{5},0)) \cup (\tfrac{F_2}{5}+(\tfrac{8}{5},0)) \cup (\tfrac{F_5}{5}+(0,\tfrac{3}{5})) \cup (\tfrac{F_9}{5}+(\tfrac{3}{5},\tfrac{3}{5})) \\
&\quad\ \cup(\tfrac{F_8}{5}+(\tfrac{8}{5},\tfrac{3}{5})) \cup (\tfrac{F_3}{5}+(0,\tfrac{8}{5})) \cup (\tfrac{F_7}{5}+(\tfrac{3}{5},\tfrac{8}{5})) \cup (\tfrac{F_6}{5}+(\tfrac{8}{5},\tfrac{8}{5})), \\
F_7 &= F \cup (F+(1,0)) \cup (F+(2,0)) \cup (F+(0,1)) \cup (F+(1,1)) \cup (F+(2,1)) \\
&= \tfrac{F}{5} \cup (\tfrac{F}{5}+(\tfrac{3}{5},0)) \cup (\tfrac{F}{5}+(\tfrac{4}{5},0)) \cup (\tfrac{F}{5}+(0,\tfrac{3}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{3}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5}+(0,\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{4}{5})) \\
&\quad \cup(\tfrac{F}{5}+(1,0)) \cup (\tfrac{F}{5}+(\tfrac{8}{5},0)) \cup (\tfrac{F}{5}+(\tfrac{9}{5},0)) \cup (\tfrac{F}{5}+(1,\tfrac{3}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{3}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5}+(1,\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{4}{5})) \\
&\quad \cup(\tfrac{F}{5}+(2,0)) \cup (\tfrac{F}{5}+(\tfrac{13}{5},0)) \cup (\tfrac{F}{5}+(\tfrac{14}{5},0)) \cup (\tfrac{F}{5}+(2,\tfrac{3}{5})) \cup (\tfrac{F}{5}+(\tfrac{13}{5},\tfrac{3}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{14}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5}+(2,\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{13}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{14}{5},\tfrac{4}{5})) \\
&\quad \cup(\tfrac{F}{5}+(0,1)) \cup (\tfrac{F}{5}+(\tfrac{3}{5},1)) \cup (\tfrac{F}{5}+(\tfrac{4}{5},1)) \cup (\tfrac{F}{5}+(0,\tfrac{8}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{8}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{8}{5})) \cup (\tfrac{F}{5}+(0,\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{9}{5})) \\
&\quad \cup(\tfrac{F}{5}+(1,1)) \cup (\tfrac{F}{5}+(\tfrac{8}{5},1)) \cup (\tfrac{F}{5}+(\tfrac{9}{5},1)) \cup (\tfrac{F}{5}+(1,\tfrac{8}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{8}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{8}{5})) \cup (\tfrac{F}{5}+(1,\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{9}{5})) \\
&\quad \cup(\tfrac{F}{5}+(2,1)) \cup (\tfrac{F}{5}+(\tfrac{13}{5},1)) \cup (\tfrac{F}{5}+(\tfrac{14}{5},1)) \cup (\tfrac{F}{5}+(2,\tfrac{8}{5})) \cup (\tfrac{F}{5}+(\tfrac{13}{5},\tfrac{8}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{14}{5},\tfrac{8}{5})) \cup (\tfrac{F}{5}+(2,\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{13}{5},\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{14}{5},\tfrac{9}{5})) \\
&= \tfrac{F_1}{5} \cup (\tfrac{F_4}{5}+(\tfrac{3}{5},0)) \cup (\tfrac{F_4}{5}+(\tfrac{8}{5},0)) \cup (\tfrac{F_2}{5}+(\tfrac{13}{5},0)) \cup (\tfrac{F_5}{5}+(0,\tfrac{3}{5})) \\
&\quad\ \cup(\tfrac{F_9}{5}+(\tfrac{3}{5},\tfrac{3}{5})) \cup (\tfrac{F_9}{5}+(\tfrac{8}{5},\tfrac{3}{5})) \cup (\tfrac{F_8}{5}+(\tfrac{13}{5},\tfrac{3}{5})) \cup (\tfrac{F_3}{5}+(0,\tfrac{8}{5})) \\
&\quad\ \cup(\tfrac{F_7}{5}+(\tfrac{3}{5},\tfrac{8}{5})) \cup (\tfrac{F_7}{5}+(\tfrac{8}{5},\tfrac{8}{5})) \cup (\tfrac{F_6}{5}+(\tfrac{13}{5},\tfrac{8}{5})), \\
F_8 &= F \cup (F+(1,0)) \cup (F+(0,1)) \cup (F+(1,1)) \cup (F+(0,2)) \cup (F+(1,2)) \\
&= \tfrac{F_1}{5} \cup (\tfrac{F_4}{5}+(\tfrac{3}{5},0)) \cup (\tfrac{F_2}{5}+(\tfrac{8}{5},0)) \cup (\tfrac{F_5}{5}+(0,\tfrac{3}{5})) \cup (\tfrac{F_9}{5}+(\tfrac{3}{5},\tfrac{3}{5})) \\
&\quad\ \cup(\tfrac{F_8}{5}+(\tfrac{8}{5},\tfrac{3}{5})) \cup (\tfrac{F_5}{5}+(0,\tfrac{8}{5})) \cup (\tfrac{F_9}{5}+(\tfrac{3}{5},\tfrac{8}{5})) \cup (\tfrac{F_8}{5}+(\tfrac{8}{5},\tfrac{8}{5})) \\
&\quad\ \cup(\tfrac{F_3}{5}+(0,\tfrac{13}{5})) \cup (\tfrac{F_7}{5}+(\tfrac{3}{5},\tfrac{13}{5})) \cup (\tfrac{F_6}{5}+(\tfrac{8}{5},\tfrac{13}{5})), \\
F_9 &= F \cup (F+(1,0)) \cup (F+(2,0)) \cup (F+(0,1)) \cup (F+(1,1)) \\
&\quad\ \cup(F+(2,1)) \cup (F+(0,2)) \cup (F+(1,2)) \cup (F+(2,2)) \\
&= \tfrac{F}{5} \cup (\tfrac{F}{5}+(\tfrac{3}{5},0)) \cup (\tfrac{F}{5}+(\tfrac{4}{5},0)) \cup (\tfrac{F}{5}+(0,\tfrac{3}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{3}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5}+(0,\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{4}{5})) \\
&\quad \cup(\tfrac{F}{5}+(1,0)) \cup (\tfrac{F}{5}+(\tfrac{8}{5},0)) \cup (\tfrac{F}{5}+(\tfrac{9}{5},0)) \cup (\tfrac{F}{5}+(1,\tfrac{3}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{3}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5}+(1,\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{4}{5})) \\
&\quad \cup(\tfrac{F}{5}+(2,0)) \cup (\tfrac{F}{5}+(\tfrac{13}{5},0)) \cup (\tfrac{F}{5}+(\tfrac{14}{5},0)) \cup (\tfrac{F}{5}+(2,\tfrac{3}{5})) \cup (\tfrac{F}{5}+(\tfrac{13}{5},\tfrac{3}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{14}{5},\tfrac{3}{5})) \cup (\tfrac{F}{5}+(2,\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{13}{5},\tfrac{4}{5})) \cup (\tfrac{F}{5}+(\tfrac{14}{5},\tfrac{4}{5})) \\
&\quad \cup(\tfrac{F}{5}+(0,1)) \cup (\tfrac{F}{5}+(\tfrac{3}{5},1)) \cup (\tfrac{F}{5}+(\tfrac{4}{5},1)) \cup (\tfrac{F}{5}+(0,\tfrac{8}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{8}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{8}{5})) \cup (\tfrac{F}{5}+(0,\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{9}{5})) \\
&\quad \cup(\tfrac{F}{5}+(1,1)) \cup (\tfrac{F}{5}+(\tfrac{8}{5},1)) \cup (\tfrac{F}{5}+(\tfrac{9}{5},1)) \cup (\tfrac{F}{5}+(1,\tfrac{8}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{8}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{8}{5})) \cup (\tfrac{F}{5}+(1,\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{9}{5})) \\
&\quad \cup(\tfrac{F}{5}+(2,1)) \cup (\tfrac{F}{5}+(\tfrac{13}{5},1)) \cup (\tfrac{F}{5}+(\tfrac{14}{5},1)) \cup (\tfrac{F}{5}+(2,\tfrac{8}{5})) \cup (\tfrac{F}{5}+(\tfrac{13}{5},\tfrac{8}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{14}{5},\tfrac{8}{5})) \cup (\tfrac{F}{5}+(2,\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{13}{5},\tfrac{9}{5})) \cup (\tfrac{F}{5}+(\tfrac{14}{5},\tfrac{9}{5})) \\
&\quad \cup(\tfrac{F}{5}+(0,2)) \cup (\tfrac{F}{5}+(\tfrac{3}{5},2)) \cup (\tfrac{F}{5}+(\tfrac{4}{5},2)) \cup (\tfrac{F}{5}+(0,\tfrac{13}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{13}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{13}{5})) \cup (\tfrac{F}{5}+(0,\tfrac{14}{5})) \cup (\tfrac{F}{5}+(\tfrac{3}{5},\tfrac{14}{5})) \cup (\tfrac{F}{5}+(\tfrac{4}{5},\tfrac{14}{5})) \\
&\quad \cup(\tfrac{F}{5}+(1,2)) \cup (\tfrac{F}{5}+(\tfrac{8}{5},2)) \cup (\tfrac{F}{5}+(\tfrac{9}{5},2)) \cup (\tfrac{F}{5}+(1,\tfrac{13}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{13}{5})) \\
&\quad\ \cup(\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{13}{5})) \cup (\tfrac{F}{5}+(1,\tfrac{14}{5})) \cup (\tfrac{F}{5}+(\tfrac{8}{5},\tfrac{14}{5})) \cup (\tfrac{F}{5}+(\tfrac{9}{5},\tfrac{14}{5}))
\end{aligned}$$

$$\begin{aligned}
&\cup(\tfrac{F}{5}+(2,2))\cup(\tfrac{F}{5}+(\tfrac{13}{5},2))\cup(\tfrac{F}{5}+(\tfrac{14}{5},2))\cup(\tfrac{F}{5}+(2,\tfrac{13}{5}))\cup(\tfrac{F}{5}+(\tfrac{13}{5},\tfrac{13}{5}))\\
&\quad\cup(\tfrac{F}{5}+(\tfrac{14}{5},\tfrac{13}{5}))\cup(\tfrac{F}{5}+(2,\tfrac{14}{5}))\cup(\tfrac{F}{5}+(\tfrac{13}{5},\tfrac{14}{5}))\cup(\tfrac{F}{5}+(\tfrac{14}{5},\tfrac{14}{5}))\\
=&\ \tfrac{F_1}{5}\cup(\tfrac{F_4}{5}+(\tfrac{3}{5},0))\cup(\tfrac{F_4}{5}+(\tfrac{8}{5},0))\cup(\tfrac{F_2}{5}+(\tfrac{13}{5},0))\cup(\tfrac{F_5}{5}+(0,\tfrac{3}{5}))\\
&\quad\cup(\tfrac{F_9}{5}+(\tfrac{3}{5},\tfrac{3}{5}))\cup(\tfrac{F_9}{5}+(\tfrac{8}{5},\tfrac{3}{5}))\cup(\tfrac{F_8}{5}+(\tfrac{13}{5},\tfrac{3}{5}))\cup(\tfrac{F_5}{5}+(0,\tfrac{8}{5}))\\
&\quad\cup(\tfrac{F_9}{5}+(\tfrac{3}{5},\tfrac{8}{5}))\cup(\tfrac{F_9}{5}+(\tfrac{8}{5},\tfrac{8}{5}))\cup(\tfrac{F_8}{5}+(\tfrac{13}{5},\tfrac{8}{5}))\cup(\tfrac{F_3}{5}+(0,\tfrac{13}{5}))\\
&\quad\cup(\tfrac{F_7}{5}+(\tfrac{3}{5},\tfrac{13}{5}))\cup(\tfrac{F_7}{5}+(\tfrac{8}{5},\tfrac{13}{5}))\cup(\tfrac{F_6}{5}+(\tfrac{13}{5},\tfrac{13}{5})).
\end{aligned}$$

For E, we consider the corresponding sets

$$\begin{aligned}
E_1 &= E,\\
E_2 &= E\cup(E+(2,0)),\\
E_3 &= E\cup(E+(0,2)),\\
E_4 &= E\cup(E+(2,0))\cup(E+(4,0)),\\
E_5 &= E\cup(E+(0,2))\cup(E+(0,4)),\\
E_6 &= E\cup(E+(2,0))\cup(E+(0,2))\cup(E+(2,2)),\\
E_7 &= E\cup(E+(2,0))\cup(E+(4,0))\cup(E+(0,2))\cup(E+(2,2))\cup(E+(4,2)),\\
E_8 &= E\cup(E+(2,0))\cup(E+(0,2))\cup(E+(2,2))\cup(E+(0,4))\cup(E+(2,4)),\\
E_9 &= E\cup(E+(2,0))\cup(E+(4,0))\cup(E+(0,2))\cup(E+(2,2))\\
&\quad\cup(E+(4,2))\cup(E+(0,4))\cup(E+(2,4))\cup(E+(4,4)).
\end{aligned}$$

From $f_i(x)=\frac{x}{5}+b_i$, $i=1,2,\cdots,9$, we can similarly obtain that

$$\begin{aligned}
E_1 &= \tfrac{E_1}{5}\cup(\tfrac{E_2}{5}+(\tfrac{2}{5},0))\cup(\tfrac{E_3}{5}+(0,\tfrac{2}{5}))\cup(\tfrac{E_6}{5}+(\tfrac{2}{5},\tfrac{2}{5})),\\
E_2 &= (\tfrac{E_1}{5}+(2,0))\cup(\tfrac{E_4}{5})\cup(\tfrac{E_2}{5}+(\tfrac{12}{5},0))\cup(\tfrac{E_3}{5}+(2,\tfrac{2}{5}))\cup(\tfrac{E_7}{5}+(0,\tfrac{2}{5}))\cup(\tfrac{E_6}{5}+(\tfrac{12}{5},\tfrac{2}{5})),\\
E_3 &= (\tfrac{E_1}{5}+(0,2))\cup(\tfrac{E_2}{5}+(\tfrac{2}{5},2))\cup(\tfrac{E_5}{5})\cup(\tfrac{E_8}{5}+(\tfrac{2}{5},0)\cup(\tfrac{E_3}{5}+(0,\tfrac{12}{5}))\cup(\tfrac{E_6}{5}+(\tfrac{2}{5},\tfrac{12}{5})),\\
E_4 &= (\tfrac{E_1}{5}+(4,0))\cup(\tfrac{E_4}{5})\cup(\tfrac{E_4}{5}+(2,0))\cup(\tfrac{E_2}{5}+(\tfrac{22}{5},0))\cup(\tfrac{E_3}{5}+(4,\tfrac{2}{5}))\\
&\quad\cup(\tfrac{E_7}{5}+(0,\tfrac{2}{5}))\cup(\tfrac{E_7}{5}+(2,\tfrac{2}{5}))\cup(\tfrac{E_6}{5}+(\tfrac{22}{5},\tfrac{2}{5})),\\
E_5 &= (\tfrac{E_1}{5}+(0,4))\cup(\tfrac{E_2}{5}+(\tfrac{2}{5},4))\cup(\tfrac{E_5}{5})\cup(\tfrac{E_8}{5}+(\tfrac{2}{5},0))\cup(\tfrac{E_5}{5}+(0,2))\\
&\quad\cup(\tfrac{E_8}{5}+(\tfrac{2}{5},2))\cup(\tfrac{E_3}{5}+(0,\tfrac{22}{5}))\cup(\tfrac{E_6}{5}+(\tfrac{2}{5},\tfrac{22}{5})),\\
E_6 &= (\tfrac{E_1}{5}+(2,2))\cup(\tfrac{E_4}{5}+(2,0))\cup(\tfrac{E_2}{5}+(\tfrac{12}{5},2))\cup(\tfrac{E_5}{5}+(0,2))\cup(\tfrac{E_9}{5})\\
&\quad\cup(\tfrac{E_8}{5}+(\tfrac{2}{5},2))\cup(\tfrac{E_3}{5}+(2,\tfrac{12}{5}))\cup(\tfrac{E_7}{5}+(2,\tfrac{2}{5}))\cup(\tfrac{E_6}{5}+(\tfrac{12}{5},\tfrac{12}{5})),\\
E_7 &= (\tfrac{E_1}{5}+(4,2))\cup(\tfrac{E_4}{5}+(4,0))\cup(\tfrac{E_4}{5}+(2,2))\cup(\tfrac{E_2}{5}+(\tfrac{22}{5},2))\cup(\tfrac{E_5}{5})\\
&\quad\cup(\tfrac{E_9}{5}+(2,0))\cup(\tfrac{E_9}{5}+(0,2))\cup(\tfrac{E_8}{5}+(\tfrac{2}{5},0))\cup(\tfrac{E_3}{5}+(4,\tfrac{12}{5}))\\
&\quad\cup(\tfrac{E_7}{5}+(4,\tfrac{2}{5}))\cup(\tfrac{E_7}{5}+(2,\tfrac{12}{5}))\cup(\tfrac{E_6}{5}+(\tfrac{22}{5},\tfrac{12}{5})),\\
E_8 &= (\tfrac{E_1}{5}+(2,4))\cup(\tfrac{E_4}{5})\cup(\tfrac{E_2}{5}+(\tfrac{12}{5},4))\cup(\tfrac{E_5}{5}+(2,2))\cup(\tfrac{E_9}{5}+(2,0))\\
&\quad\cup(\tfrac{E_8}{5}+(\tfrac{12}{5},2))\cup(\tfrac{E_5}{5}+(0,4))\cup(\tfrac{E_9}{5}+(0,2))\cup(\tfrac{E_8}{5}+(\tfrac{2}{5},4))\\
&\quad\cup(\tfrac{E_3}{5}+(2,\tfrac{22}{5}))\cup(\tfrac{E_7}{5}+(0,\tfrac{2}{5}))\cup(\tfrac{E_6}{5}+(\tfrac{12}{5},\tfrac{22}{5})),\\
E_9 &= (\tfrac{E_1}{5}+(4,4))\cup(\tfrac{E_4}{5}+(2,0))\cup(\tfrac{E_4}{5}+(0,2))\cup(\tfrac{E_2}{5}+(\tfrac{22}{5},4))\cup(\tfrac{E_5}{5})\\
&\quad\cup(\tfrac{E_9}{5}+(4,0))\cup(\tfrac{E_9}{5}+(0,4))\cup(\tfrac{E_8}{5}+(\tfrac{2}{5},0))\cup(\tfrac{E_5}{5}+(2,2))\\
&\quad\cup(\tfrac{E_9}{5}+(4,2))\cup(\tfrac{E_9}{5}+(2,4))\cup(\tfrac{E_8}{5}+(\tfrac{12}{5},2))\cup(\tfrac{E_3}{5}+(4,\tfrac{22}{5}))\\
&\quad\cup(\tfrac{E_7}{5}+(2,\tfrac{2}{5}))\cup(\tfrac{E_7}{5}+(0,\tfrac{12}{5}))\cup(\tfrac{E_6}{5}+(\tfrac{22}{5},\tfrac{22}{5})).
\end{aligned}$$

It is easy to see that both $\{F_i\}_{i=1}^{9}$ and $\{E_i\}_{i=1}^{9}$ are dust-like. Since the base graphs coincide and all the similitudes have ratio $\frac{1}{5}$, Theorem 2.1 shows that $E_1 \simeq F_1$, i.e., $E \simeq F$. $\square$

4 Conclusion

In this paper, we mainly study the Lipschitz equivalence of two Sierpinski gaskets. In section 1, we recall the definition of Lipschitz equivalence of two sets and introduce some known results about this subject. In section 2, we recall the definition of graph-directed sets and related results which play an important role in Theorem 3.2. Our main work of this paper is to study the Lipschitz equivalence of two Sierpinski gaskets with the same ratios. The main result is given in Theorem 3.2.

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