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The Existence of Almost Periodic Solutions of Several Classes of Second Order Differential Equations

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Abstract: In this paper, we deal with the existence of almost periodic solutions of the second-order differential equations $x'' + f(x) + g(x(t-r)) = e(t)$ and $x'' + f(t,x) = e(t)$ with exponential dichotomy theories. By using exponential dichotomy theories, we have proved that the above equations have at least one almost periodic solution in some given conditions. Moreover, we obtain the sufficient conditions which guarantee the existence of unstable almost periodic solutions of Hill equation and Mathien equation respectively.

Keywords: exponential Dichotomy; almost periodic solutions; existence

1 Introduction

In research works, certain equations are often considered by scholars. Ding, Tian and Lu[1,2] studied dissipative Camassa-Holm Equation and obtained the existence and uniqueness of the stationary solution belonging to absorbing set; G.A.Afrouzi1 et al[3] obtained the existence of solutions to a non-autonomous p-Laplacian equation; Fan et al.[4] investigated multiple compactons in a nonlinear atomic chain equation.

In this paper, we consider the following Duffing equation

$$x'' + f(x) = e(t),$$

the equation above is widely studied by many scholars, mainly about the existence of periodic solutions(see [5-10]).

However, few scholars studied the existence of almost periodic solutions of the following Duffing-type equation (1.1) and the equation (1.2).

$$x'' + f(x) + g(x(t-r)) = e(t). \tag{1.1}$$

$$x'' + f(t,x) = e(t). \tag{1.2}$$

In [11-13], the method of nonlinear variable change was used widely, by analyzing the equivalent differential equation which posses an almost periodic solution, thereby the almost periodic solution of initial differential equation was obtained. Stimulated by the works of [11-13], in this paper, we use variable change, theories of exponential dichotomy to study the Duffing-type equations (1.1) and (1.2) and discuss the existence of almost periodic solutions of Duffing-type equations with delay and without delay, furthermore, we investigate Hill equation and Mathien equation and obtain the existence of unstable unique almost periodic solutions for the two kinds of differential equations.

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2 Preliminary definition and lemmas

Definition 2.1([14],P72) Consider the linear differential equation

$$x' = A(t)x, \tag{2.1}$$

where the coefficient matrix $A(t)$ is continuous on an interval J. The equation (2.1) is said to be kinematically similar to another equation(denote by $A(t) \cong B(t)$)

$$y' = B(t)y, \tag{2.2}$$

if there exists a continuously differentiable invertible matrix $S(t)$ which satisfies the differential equation

$$S'(t) = A(t)S(t) - S(t)B(t), \tag{2.3}$$

and which is bounded, together with its inverse, on J. The change of variables $x = S(t)y$ then transforms (2.1) into (2.2).

Lemma 2.1([11],P72) If linear system (2.1) posses an exponential dichotomy with projection P, and $A(t) \cong B(t)$, then linear system (2.2) also posses an exponential dichotomy with projection Q, and the projection Q has the same nullspace as the projection P .

Lemma 2.2 Consider the following system

$$x' = A(t,x)x + f(t). \tag{2.4}$$

where $A \in C(R \times R^n, R^n)$ is an $n \times n$ continuous almost periodic matrix function in t uniformly with respect to $x \in R^n$, and $f(t) \in C(R, R^n)$ is an almost periodic vector function in t . If for any bounded continuous function $\varphi(t)$ such that the system

$$x' = A(t, \varphi(t))x \tag{2.5}$$

has an exponential dichotomy, namely, the fundamental matrix $X_\varphi(t)$ of the system (2.5) satisfies

$$\|X_\varphi(t)PX_\varphi^{-1}(s)\| \leq \beta\exp(-\alpha(t-s)), (t \geq s)$$

$$\|X_\varphi(t)(I-P)X_\varphi^{-1}(s)\| \leq \beta\exp(-\alpha(s-t)), (t \leq s)$$

where P is a projection, β and α are positive constants, then there exists an almost periodic solution of the equation (2.4).

Proof Let $B = \{g(t)|g$ is any continuous almost periodic function$\}$, the norm is defined as $\|g\| = \sup_{t\in R}\|g(t)\|$, so $(B, \|\cdot\|)$ is a Banach space. Note that $g(t)$ is a continuous almost periodic function, thus $g(t)$ is bounded, and also $f(t)$ is almost periodic. One can select a positive constant R_0 such that $\sup_{t\in R}\|g(t)\| \leq R_0$ and $\frac{1}{R_0}\|f(t)\| \leq \frac{\alpha}{2\beta}$. Constructing a compact subset of B as follows:

$$B_0 = \{g(t)|g \in B, \|g\| \leq R_0\},$$

for any $\varphi(t) \in B_0$, consider the nonhomogeneous system as follows:

$$x' = A(t, \varphi(t))x + f(t). \tag{2.6}$$

Since $\varphi(t) \in B_0$, and $\varphi(t)$ is bounded, according to the condition of Lemma 2.1, the linear system

$$x' = A(t, \varphi(t))x \tag{2.7}$$

has an exponential dichotomy. Since $\varphi(t)$ is almost periodic, and $A(t, x)$ is almost periodic in t uniformly with respect to x, $A(t, \varphi(t))$ is almost periodic.Thus the system (2.6) is an almost periodic system. From Coppel [14,Proposition 8.3], there exists an almost periodic solution of the equation (2.6) as follows:

$$x_\varphi(t) = \int_{-\infty}^{t} X_\varphi(t)PX_\varphi^{-1}(s)f(s)ds - \int_{t}^{+\infty} X_\varphi(t)(I-P)X_\varphi^{-1}(s)f(s)ds. \tag{2.8}$$

Note that

$$\begin{aligned}\|x_\varphi(t)\| &= \|\int_{-\infty}^{t} X_\varphi(t)PX_\varphi^{-1}(s)f(s)ds - \int_{t}^{+\infty} X_\varphi(t)(I-P)X_\varphi^{-1}(s)f(s)ds\| \\ &\le (\int_{-\infty}^{t} \|X_\varphi(t)PX_\varphi^{-1}(s)\|ds + \int_{t}^{+\infty} \|X_\varphi(t)(I-P)X_\varphi^{-1}(s)\|ds) \sup_{t\in R}\|f(t)\| \\ &\le (\int_{-\infty}^{t} \beta e^{-\alpha(t-s)}ds + \int_{t}^{+\infty} \beta e^{-\alpha(s-t)}ds) \sup_{t\in R}\|f(t)\| \\ &= \frac{2\beta}{\alpha}\|f\| \\ &\le R_0\end{aligned}$$

This means that $x_\varphi(t) \in B_0$. Now define mapping $T : B_0 \to B_0, T_\varphi = x_\varphi$. Suppose that the sequence $\{\varphi_n(t)\} \subseteq B_0, \|T_{\varphi_n}(t)\| \le R_0, T_{\varphi_n}(t) = x_{\varphi_n}(t)(n = 1, 2, \cdots)$ satisfy

$$\frac{dx_{\varphi_n}(t)}{dt} = A(t, \varphi_n(t))x_{\varphi_n}(t) + f(t), \tag{2.9}$$

it follows

$$\|\frac{dx_{\varphi_n}(t)}{dt}\| \le (M + \frac{\alpha}{2\beta})R_0, \tag{2.10}$$

where $M = \sup_{t\in R,\|\varphi_n(t)\|\le R_0} \|A(t, \varphi_n(t))\|$. This means that $\|\frac{dx_{\varphi_n}(t)}{dt}\|$ is bounded uniformly, thus $\{x_{\varphi_n}(t)\}$ is bounded uniformly and equicontinuous. From Ascoli's Theorem, there exists a subsequence $\{x_{\varphi_{n_k}}(t)\}$ of $\{x_{\varphi_n}(t)\}$ such that the subsequence $\{x_{\varphi_{n_k}}(t)\}$ converges uniformly in any compact set of R. Noticing that $x_{\varphi_{n_k}}(t) = T_{\varphi_{n_k}}(t)$, therefore, the sequence $\{T_{\varphi_{n_k}}(t)\}$ is also convergent uniformly on R.

Next we shall prove that T is a continuous mapping. Suppose that the sequence $\{\varphi_n(t)\} \subseteq B_0$, and $\varphi_n(t) \to \varphi(t)$, then $X_{\varphi_n}(t) \to X_\varphi(t)$ converges uniformly on any compact set of R, where $X_{\varphi_n}(t)$ is the fundamental matrix of the following linear system

$$x'(t) = A(t, \varphi_n(t))x(t). \tag{2.11}$$

Since the fundamental matrix $X_{\varphi_n}(t)$ satisfies the exponential dichotomy, thus, the following infinity integrals

$$\int_{-\infty}^{t} X_{\varphi_n}(t)PX_{\varphi_n}^{-1}(s)f(s)ds$$

and

$$-\int_{t}^{+\infty} X_{\varphi_n}(t)(I-P)X_{\varphi_n}^{-1}(s)f(s)ds$$

are convergent uniformly in n. From Coppel[14,Proposition 5.1] and

$$x_{\varphi_n}(t) = \int_{-\infty}^{t} X_{\varphi_n}(t)PX_{\varphi_n}^{-1}(s)f(s)ds - \int_{t}^{+\infty} X_{\varphi_n}(t)(I-P)X_{\varphi_n}^{-1}(s)f(s)ds, \tag{2.12}$$

we know that $x_{\varphi_n}(t)$ converges uniformly to $x_\varphi(t)$, namely, $T_{\varphi_n} \to T_\varphi$, Therefore, T is a continuous mapping. By Schauder's fixed point theorem, there exists a fixed point of T, which implies that there exists a $\varphi \in B_0$ such that $T_\varphi = \varphi$, that is to say, there exists an almost periodic solution of the equation (2.4).

Lemma 2.3([11]): If the coefficient matrix $A(t)$ of the system (2.1) is real and satisfied row dominance or column dominance of the following conditions

(I)

$$a_{ii}(t) + \sum_{j\ne i}|a_{ij}(t)| \le -\delta < 0, i = 1, 2, \cdots, k,$$

$$a_{ii}(t) - \sum_{j\ne i}|a_{ij}(t)| \ge \delta > 0, i = k+1, k+2, \cdots, n$$

(II)

$$a_{jj}(t)+\sum_{i\neq j}|a_{ij}(t)|\leq -\delta<0, j=1,2,\cdots,k,$$

$$a_{jj}(t)-\sum_{i\neq j}|a_{ij}(t)|\geq \delta>0, j=k+1,k+2,\cdots,n,$$

then the system (2.1) has an exponential dichotomy with projection $P=\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$, dichotomy constants $(\delta,1)$.

Lemma 2.4 If the conditions of Lemma 2.3 hold, and $k<n$, then the zero solution of the system (2.1) is unstable.

Proof Suppose $X(t)$ is a fundamental matrix solution of the system (2.1) which satisfies $X(0)=I$, $x(t)$ is the non-trivial solution of the system (2.1) with initial value $(0,\xi)$(where ξ is an arbitrary nonzero constant vector which at least one value from one to k dimensional subspace isn't identically 0, and at least one value from $k+1$ to n dimensional subspace is not identically 0). From the condition, by Lemma 2.3, we know the system (2.1) has an exponential dichotomy with projection $P=\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$, where $k<n$, dichotomy constants $(\delta,1)$, hence we have:

$$\|X(t)PX^{-1}(s)\|\leq e^{-\delta(t-s)},(t\geq s) \tag{2.13}$$

$$\|X(t)(I-P)X^{-1}(s)\|\leq e^{-\delta(s-t)},(s\geq t) \tag{2.14}$$

multiplying both sides of the inequality (2.13) by $\|X(s)P\xi\|$ gets

$$\|X(t)PX^{-1}(s)X(s)P\xi\|\leq \|X(s)P\xi\|e^{-\delta(t-s)},(t\geq s)$$

that is

$$\|X(t)P^2\xi\|\leq \|X(s)P\xi\|e^{-\delta(t-s)},(t\geq s)$$

since $P^2=P$, we have

$$\|X(t)P\xi\|\leq \|X(s)P\xi\|e^{-\delta(t-s)},(t\geq s) \tag{2.15}$$

Let $s=0$, and that $X(0)=I$, it follows

$$\|X(t)P\xi\|\leq \|P\xi\|e^{-\delta t},(t\geq 0) \tag{2.16}$$

Note that $P\xi\neq 0$, from (2.15) and (2.16), we know, in k dimensional subspace , $\|X(t)P\xi\|$ is a strictly descending function , as $t\to+\infty$,$\|X(t)P\xi\|\to 0$.

Multiplying both sides of the the inequality (2.14) by $\|X(s)(I-P)\xi\|$ gets

$$\|X(t)(I-P)X^{-1}(s)X(s)(I-P)\xi\|\leq \|X(s)(I-P)\xi\|e^{-\delta(s-t)},(s\geq t)$$

that is

$$\|X(t)(I-P)^2\xi\|\leq \|X(s)(I-P)\xi\|e^{-\delta(s-t)},(s\geq t)$$

since $(I-P)^2=I-P$, we have

$$\|X(t)(I-P)\xi\|\leq \|X(s)(I-P)\xi\|e^{-\delta(s-t)},(s\geq t) \tag{2.17}$$

Let $t=0$,and note that $X(0)=I$, then we have

$$\|(I-P)\xi\|\leq \|X(s)(I-P)\xi\|e^{-\delta s},(s\geq 0)$$

thus

$$\|X(s)(I-P)\xi\| \geq \|(I-P)\xi\| e^{\delta s}.(s \geq 0) \tag{2.18}$$

Since $P \neq I$, $\xi \neq 0$, $(I-P)\xi \neq 0$, from (2.17) and (2.18), we know, in $n-k$ dimensional subspace, $\|X(t)(I-P)\xi\|$ is a strictly increasing function, as $t \to +\infty$, $\|X(t)(I-P)\xi\| \to +\infty$.
Hence in n dimensional space, the solution of system (2.1) holds as follows:
$x(t) = X(t)\xi = X(t)P\xi + X(t)(I-P)\xi,$
so we have

$$\|x(t)\| = \|X(t)\xi\| = \|X(t)P\xi + X(t)(I-P)\xi\| \geq \|X(t)(I-P)\xi\| - \|X(t)P\xi\|, \tag{2.19}$$

from (2.16),(2.18),(2.19), as $t \to +\infty$, $\|x(t)\| \to +\infty$. Hence the zero solution of system (2.1) is unstable.

Lemma 2.5 Consider the following equation

$$x'(t) = A(t)x(t) + f(t), \tag{2.20}$$

where $A(t)$ is a continuous matrix function, and $f(t)$ is a continuous vector function. If the zero solution of the homogeneous equation of the system (2.20) is unstable, then every solution of the system (2.20) is unstable.

3 The existence of almost periodic solutions of Duffing equations

In this section, we discuss Duffing equations, and obtain two Theorems which guarantee the existence of almost periodic solutions for the equation (1.1) and (1.2) respectively.
Theorem 3.1 Consider the equation (1.1). r is a positive real-valued constant. $g(\phi) \in C(R,R)$ is a continuous function in ϕ. $f(x) \in C(R,R)$ is a differentiable function in x. $e(t) \in C(R,R)$ is an almost periodic function in t. If the following conditions hold: (1)$f(0) \equiv 0$; (2) $-\infty < \inf_{x\in R} f'(x) \leq \sup_{x\in R} f'(x) < 0$; (3) $-\infty < \inf_{\phi\in R} g(\phi) \leq \sup_{\phi\in R} g(\phi) < +\infty$, then there exists an almost periodic solution of the equation (1.1).
Proof Take a change of variables as follows:

$$\begin{cases} u = x, \\ v = \lambda(x' - bx), \end{cases}$$

where λ, b are constants to be determined later. Then the equation (1.1) is changed to the following form:

$$\begin{cases} \dot{u} = bu + \frac{1}{\lambda}v, \\ \dot{v} = (-\lambda K(u) - \lambda b^2)u - bv + \lambda e(t) - g(u(t-r)), \end{cases} \tag{3.1}$$

where

$$K(u) = \begin{cases} u^{-1}f(u), u \neq 0, \\ f'(0), u = 0. \end{cases}$$

Let

$$A(u) = \begin{pmatrix} b & \frac{1}{\lambda} \\ -\lambda K(u) - \lambda b^2 & -b \end{pmatrix},$$

and

$$h(t, u(t-r)) = \begin{pmatrix} 0 \\ \lambda e(t) - g(u(t-r)) \end{pmatrix},$$

then the equation (3.1) can be rewritten as follows:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A(u) \begin{pmatrix} u \\ v \end{pmatrix} + h(t, u(t-r)). \tag{3.2}$$

Let $c_1 = \inf_{x\in R} f'(x)$, and $c_2 = \sup_{x\in R} f'(x)$, obviously $c_1 \leq c_2$.
Assume that $b < 0$. If there is a $\delta > 0$ such that

$$\begin{cases} b + |\lambda||K(u) + b^2| \leq -\delta, \\ -b - |\frac{1}{\lambda}| \geq \delta. \end{cases} \tag{3.3}$$

Let $\delta < -b$ and $|\lambda| = -\frac{1}{b+\delta}$, then the equation (3.3) is equivalent to

$$\begin{cases} 0 < \delta < -b, b < 0, \\ -2b^2 - \delta^2 - 2b\delta \leq K(u) \leq \delta^2 + 2b\delta. \end{cases} \tag{3.4}$$

Since $c_2 < 0$, obviously $c_1 < 0$, let $b = -\delta - \frac{\sqrt{-c_1}}{\sqrt{2}}$, then $b + \delta < 0$, $\delta < -b$ and $2b^2 + c_1 = 2\delta^2 + 2\sqrt{2}\delta\sqrt{-c_1} > 0$.
Note that when $c_2 < 0$, $2b^2 + c_1 > 0$, there must be a smaller constant $\delta > 0$ such that

$$-2b^2 - \delta^2 - 2b\delta \leq c_1 \leq K(u) \leq \delta^2 + 2b\delta.$$

So the inequalities (3.4) hold. By Lemma 2.3, we know

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A(u) \begin{pmatrix} u \\ v \end{pmatrix}$$

has an exponential dichotomy. Notice that $g(u(t-r))$ is a bounded function, and $e(t)$ is an almost periodic function, so $e(t)$ is bounded. λ is also a bounded constant, therefore, we know, $\sup_{(t,u,v)\in R^3} |h(t,u,v)| < +\infty$. For any bounded function $\varphi(t)$, there is an exponential dichotomy of the following system:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A(\varphi(t)) \begin{pmatrix} u \\ v \end{pmatrix}.$$

Since $h(t,u,v)$ is bounded and almost periodic in t, from the Lemma 1 of [13], there is an almost periodic solution of system (3.2). This means that there is an almost periodic solution of the equation (1.1).

Theorem 3.2 Consider the equation (1.2), $f(t,x) \in C(R \times R, R)$ is continuous almost periodic function in t and differentiable with respect to $x \in R$, $e(t) \in C(R,R)$ is an almost periodic function. If the following conditions hold: (1)$f(t,0) \equiv 0$ for every $t \in R$, (2) $-\infty < \inf_{(t,x)\in(R\times R)} f'_x(t,x) \leq \sup_{(t,x)\in(R\times R)} f'_x(t,x) < 0$, then there exists an almost periodic solution of the equation (1.2).
Proof Take a change of variables as follows:

$$\begin{cases} u = x, \\ v = \lambda(x' - bx), \end{cases} \tag{3.5}$$

where λ, b are constants to be determined late. Then the equation (1.2) is changed to the following form:

$$\begin{cases} \dot{u} = bu + \frac{1}{\lambda}v, \\ \dot{v} = (-\lambda K(t,u) - \lambda b^2)u - bv + \lambda e(t), \end{cases} \tag{3.6}$$

where

$$K(t,u) = \begin{cases} u^{-1}f(t,u), u \neq 0, \\ f'_u(t,0), u = 0. \end{cases}$$

Let

$$A(t,u) = \begin{pmatrix} b & \frac{1}{\lambda} \\ -\lambda K(t,u) - \lambda b^2 & -b \end{pmatrix},$$

and

$$h(t) = \begin{pmatrix} 0 \\ \lambda e(t) \end{pmatrix},$$

then the equation (3.6) can be rewritten as follows:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A(t,u) \begin{pmatrix} u \\ v \end{pmatrix} + h(t). \tag{3.7}$$

Let $c_1 = \inf_{(t,x)\in(R\times R)} f'(t,x), c_2 = \sup_{(t,x)\in(R\times R)} f'(t,x)$, obviously $c_1 \le c_2$.
Assume that $b < 0$, if there is a $\delta > 0$ such that

$$\begin{cases} b + |\lambda||K(t,u) + b^2| \le -\delta, \\ -b - |\frac{1}{\lambda}| \ge \delta. \end{cases} \tag{3.8}$$

Let $\delta < -b$ and $|\lambda| = -\frac{1}{b+\delta}$, then the inequalities (3.8) are equivalent to

$$\begin{cases} 0 < \delta < -b, b < 0, \\ -2b^2 - \delta^2 - 2b\delta \le K(t,u) \le \delta^2 + 2b\delta. \end{cases} \tag{3.9}$$

Since $c_2 < 0$, obviously $c_1 < 0$, let $b = -\delta - \frac{\sqrt{-c_1}}{\sqrt{2}}$,then $b + \delta < 0$,$\delta < -b$ and $2b^2 + c_1 = 2\delta^2 + 2\sqrt{2}\delta\sqrt{-c_1} > 0$.
Note that when $c_2 < 0$, $2b^2 + c_1 > 0$, there must be a smaller constant $\delta > 0$ such that

$$-2b^2 - \delta^2 - 2b\delta \le c_1 \le K(t,u) \le \delta^2 + 2b\delta$$

So the inequalities (3.9) hold. By Lemma 2.3, we know

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A(t,u) \begin{pmatrix} u \\ v \end{pmatrix} \tag{3.10}$$

has an exponential dichotomy.

From the conditions of Theorem 3.2, $f(t,x)$ is continuous almost periodic in t uniformly with respect to $x \in R$, thus $A(t,u)$ is continuous almost periodic in t and uniformly with respect to $u \in R$. In addition, notice that $e(t)$ is an almost periodic function, so $\lambda e(t)$is also almost periodic. Hence $h(t)$ is bounded and almost periodic function in t. According to the Lemma 2.2, there is an almost periodic solution of the system (3.7).This means that there is an almost periodic solution of the system (1.2).

4 The existence of unstable almost periodic solution of Hill equation

In this section, we discuss Hill equation

$$x'' + H(t)x = e(t), \tag{4.1}$$

where $H(t), e(t)$ are both continuous almost periodic functions.
Theorem 4.1 Consider the equation (4.1). Suppose that $-\infty < \inf_{t\in R} H(t) \le \sup_{t\in R} H(t) < 0$, then there exists a unique almost periodic solution of the equation (4.1) which is unstable.

Proof Take a change of variables as follows:

$$\begin{cases} u = x, \\ v = \lambda(x' - bx), \end{cases} \tag{4.2}$$

where λ, b are constants to be determined later.Then the equation (1.2) is changed to the following form:

$$\begin{cases} \dot{u} = bu + \frac{1}{\lambda}v, \\ \dot{v} = (-\lambda H(t) - \lambda b^2)u - bv + \lambda e(t), \end{cases} \tag{4.3}$$

Let

$$A(t) = \begin{pmatrix} b & \frac{1}{\lambda} \\ -\lambda H(t) - \lambda b^2 & -b \end{pmatrix},$$

and

$$f(t) = \begin{pmatrix} 0 \\ \lambda e(t) \end{pmatrix},$$

then the equation (4.3) can be rewritten as follows:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A(t) \begin{pmatrix} u \\ v \end{pmatrix} + f(t). \tag{4.4}$$

Let $c_1 = \inf_{t \in R} H(t), c_2 = \sup_{t \in R} H(t)$, obviously $c_1 \leq c_2$.
Assume that $b < 0$, if there is a $\delta > 0$ such that

$$\begin{cases} b + |\lambda||H(t) + b^2| \leq -\delta, \\ -b - |\frac{1}{\lambda}| \geq \delta, \end{cases} \tag{4.5}$$

Let $\delta < -b$ and $|\lambda| = -\frac{1}{b+\delta}$.Then the inequalities (4.5) are equivalent to

$$\begin{cases} 0 < \delta < -b, b < 0, \\ -2b^2 - \delta^2 - 2b\delta \leq H(t) \leq \delta^2 + 2b\delta, \end{cases} \tag{4.6}$$

Since $c_2 < 0$,obviously $c_1 < 0$, let $b = -\delta - \frac{\sqrt{-c_1}}{\sqrt{2}}$,then $b+\delta < 0$,$\delta < -b$ and $2b^2+c_1 = 2\delta^2+\sqrt{2}\delta\sqrt{-c_1} > 0$. Note that when $c_2 < 0$, $2b^2 + c_1 > 0$, there must be a smaller constant $\delta > 0$ such that

$$-2b^2 - \delta^2 - 2b\delta \leq c_1 \leq H(t) \leq \delta^2 + 2b\delta,$$

so the inequalities (4.6) hold. By Lemma 2.3, we know

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A(t) \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.7}$$

has an exponential dichotomy. In addition, notice that $e(t)$ is an almost periodic function, so $\lambda e(t)$ is also almost periodic, hence $f(t)$ is bounded and almost periodic function in t. According to the Theorem 3.4([15,P93]), there is a unique almost periodic solution of the system (4.3). This means that there is a unique almost periodic solution of the system (4.1).
Next we will prove that the unique almost periodic solution of the equation (4.1) is unstable.
The equation (4.1) is equivalent to the following form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -H(t) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ e(t) \end{pmatrix} \tag{4.8}$$

From (4.2), it follows

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\lambda b & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.9}$$

Let

$$B(t) = \begin{pmatrix} 0 & 1 \\ -H(t) & 0 \end{pmatrix}, S = \begin{pmatrix} 1 & 0 \\ -\lambda b & \lambda \end{pmatrix} \tag{4.10}$$

Next, we will prove that the change of (4.2) is Lyapunov transformation.
By calculating, we can get

$$A(t)S - SB(t) = 0.$$

Clearly, the following equality

$$S' = A(t)S - SB(t)$$

holds. By Definition 2.1, $A(t) \cong B(t)$. By Lemma 2.1 , it follows that the linear system of (4.8) admits an exponential dichotomy with projection $P \neq I$. Thus the system (4.8) exists a unique almost periodic solution. By Lemma 2.4 and 2.5, the almost periodic solution of the system (4.8) is unstable, by Lemma 2.1, 2.4 and 2.5, the almost periodic solution of the system (4.1) is also unstable. This is the end of Theorem 4.1.

From Theorem 4.1, we can obtain Theorem 4.2 immediately.
Theorem 4.2 Consider the following Mathieu differential equation

$$x'' + (\alpha + \beta cost)x = e(t), \tag{4.11}$$

where α, β are real-valued constants, and $e(t) \in C(R, R)$ is an almost periodic function. If the following conditions hold: $-\infty < \inf_{t\in R} \alpha + \beta cost \le \sup_{t\in R} \alpha + \beta cost < 0$, then there exists a unique unstable almost periodic solution of the equation (4.11).

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