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Variational Methods for the Four-body Problems

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Abstract: For Newtonian four-body problems with equal masses, we prove the existence of noncollision periodic solution such that four bodies move on a three rose curves by variational methods.

Key words: periodic solutions; four-body problems; variational methods.

1 Introduction

In recent years, many authors(for example, [1-3,5-17,19-20,22-25]) used variational methods to study the periodic solutions for Newtonian n–body problems. Especially, Chenciner–Montgomery [8] proved the existence of the remarkable figure- ”8” type periodic solution for planar Newtonian 3–body problems with equal masses, and Simo [21] used computer to discover many new periodic solutions for newtonian n–body problems.In this paper we prove the existence of the three rose solutions for planar Newtonian 4–body problems with equal masses $m_1 = m_2 = m_3 = m_4 = 1$ and $T = 1$.

The classical 4-body describes the motion of 4 punctual masses under the action of Newton's gravitation law of attraction.Let $q_i \in R^d, i = 1, \cdots, 4$ the positions of the bodies and $m_j \geq 0, j = 1, \cdots, 4$ the respective masses.In this paper we shall consider the planar problem $d = 2$. The motion of 4 masses in the plane is described by the following nonlinear system of second order differential equations:

$$\ddot{q}_i + \frac{\partial V(q)}{\partial q_i} = 0 \quad i = 1, \cdots, 4, \tag{1.1}$$

where

$$V(q) = V(q_1, \cdots, q_4) = -\sum_{1\leq i<j\leq 4} \frac{1}{|q_i - q_j|}.$$

We define the loop space by

$$E = \{q = (q_1, \cdots, q_4)|q_i \in W^{1,2}(R/Z, R^2), q_i \neq q_j \text{ for all } i \neq j \text{ and} \sum_{i=1}^{4} q_i = 0\},$$

where

$$W^{1,2}(R/Z, R^2) = \{x|x \text{ is absolutely continuous,}$$
$$x(t + 1) = x(t), \dot{x} \in L^2(R, R^2)\},$$

and the Lagrangian action on E is defined by

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$$f(q) = f(q_1, \cdots, q_4) = \int_0^1 (\frac{1}{2}\sum_{i=1}^{4} |\dot{q}_i|^2 - V(q))dt.$$

It is known that the critical points of the Lagrange functional in E are non-collision periodic solutions to (1.1).

A periodic solution to (1.1) is called choreography if all four masses travel along the same closed curve and staggered in phase from each other by $1/4$, i.e. $q_i(t + 1/4) = q_{i+1}(t),\ \ i = 1, 2, 3, 4$. We define the space of choreography by

$$\Lambda = \{q \in E | \mathrm{O}(\frac{2\pi}{3})q_{i+1}(t + \frac{1}{12}) = q_i(t), i = 1, 2, 3, \mathrm{O}(\frac{2\pi}{3})q_1(t + \frac{1}{12}) = q_4(t),$$

$$q_1(t) = Bq_1(-t), q_3(t) = Bq_3(-t), q_2(t) = Bq_4(-t), q_4(t) = Bq_2(-t)\},$$

where

$$B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Theorem 1.1 The global minimizer of $f(q)$ on $\overline{\Lambda}$ is a non-collision periodic solution of (1.1)

2 Some lemmas

Lemma 2.1 ([20]) Let X be a reflexive Banach space and $M \subset X$ a weakly closed subset. $f : M \to R$ is weakly lower semi–continuous. if f is coercive, that is, $f(x) \to +\infty$ as $||x|| \to +\infty$, then f attains its infimum on M.

Lemma 2.2 (Palais [18]) Let G be an orthogonal group on a Hilbert space H.

Define the fixed point space: $F_G = \{x \in H | gox = x, \forall g \in G\}. If\ f \in C^1(H, R)$ and satisfies $f(gox) = f(x)$ for any $g \in G$ and $x \in H$, then the critical point of f restricted on F_G is also a critical point of f on H.

Lemma 2.3(Gordon's theorem, [13])

(1) Let $x \in W^{1,2}([t_1, t_2], R^K)$ and $x(t_1) = x(t_2) = 0$. Then for any $a > 0$, we have

$$\int_{t_1}^{t_2} (\frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|})dt \geq \frac{3}{2}(2\pi)^{2/3} \cdot a^{2/3}(t_2 - t_1)^{1/3}.$$

(2) Let $x \in W^{1,2}(R/TZ, R^K)$, then for any $a > 0$, we have

$$\int_0^T (\frac{1}{2}|\dot{x}|^2 + \frac{a}{|x|})dt \geq \frac{3}{2}(2\pi)^{2/3} a^{2/3} T^{1/3}.$$

3 The proofs of Theorem 1.1

Define a group action $G = \sigma_1 \times \sigma_2$:

$$\sigma_1(q_1(t), q_2(t), q_3(t), q_4(t)) =$$

$$(O(\frac{2\pi}{3})q_2(t + \frac{1}{12}), O(\frac{2\pi}{3})q_3(t + \frac{1}{12}), O(\frac{2\pi}{3})q_4(t + \frac{1}{12}), O(\frac{2\pi}{3})q_1(t + \frac{1}{12})),$$

$$\sigma_2(q_1(t), q_2(t), q_3(t), q_4(t)) = (Bq_1(-t), Bq_4(-t), Bq_3(-t), Bq_2(-t)).$$

This implies that Λ is the fixed point space of G. Furthermore, for any σ_i and $q = (q_1, \cdots, q_4) \in (W^{1,2}(R/Z, R^2))^4$ and satisfying $q_i(t) \neq q_j(t)$ for $i \neq j$ and $t \in [0, 1]$, we have $f(\sigma_i \circ q) = f(q)$, then

Palais's symmetry principle implies the critical point of f restricted on Λ is also a critical point of f on $(W^{1,2}(R/Z,R^2)^4$ and satisfying $q_i(t)\neq q_j(t)$ for $i\neq j$ and $t\in[0,1]$.

Proposition 3.1 For any $q=(q_1(t),q_2(t),q_3(t),q_4(t))\in\Lambda$, we have

$$\int_0^1 q_i(t)dt=0.$$

Proof. For any $q=(q_1(t),q_2(t),q_3(t),q_4(t))\in\Lambda$, we have

$$\sum_{i=1}^4 q_i=0.$$

So

$$\int_0^1 q_i(t)dt=\int_0^{1/4}q_i(t)dt+\int_{1/4}^{1/2}q_i(t)dt+\int_{1/2}^{3/4}q_i(t)dt+\int_{3/4}^{1}q_i(t)dt$$
$$=\int_0^{1/4}\sum_{j=1}^4 q_jdt=0.$$

Then Poincare-Wirtinger inequality implies

$$\int_0^1|\dot q_i|^2dt\geq(2\pi)^2\int_0^1|q_i(t)|^2dt.$$

Hence $f(q)$ is coercive on $\overline{\Lambda}$. It's easy to see $\overline{\Lambda}$ is a weakly closed subset. Fatou's lemma implies that $f(q)$ is weakly lower semi-continuous. Then by Lemma 2.1, $f(q)$ attains $inf\{f(q)|q\in\overline{\Lambda}\}$.

Proposition 3.2 The minimizer of $f(q)$ on $\overline{\Lambda}$ is non-collision.

Proof. Firstly, we estimate the infimum of the action functional on the collision set.

1°. If q_1,q_2 collide at $t=0$, then q_1,q_2 collide at $t=0$ and $t=1/12$ and $t=1/3$ and $t=5/12$ and $t=2/3$ and $t=3/4$; q_1,q_4 collide at $t=0$ and $t=1/4$ and $t=1/3$ and $t=7/12$ and $t=2/3$ and $t=11/12$;q_2,q_3 collide at $t=1/12$ and $t=1/6$ and $t=5/12$ and $t=1/2$ and $t=3/4$ and $t=5/6$; q_3,q_4 collide at $t=1/6$ and $t=1/4$ and $t=1/2$ and $t=7/12$ and $t=5/6$ and $t=11/12$.

2°. If q_1,q_3 collide at $t=0$ then q_1,q_3 collide at $t=0$ and $t=1/6$ and $t=1/3$ and $t=1/2$ and $t=2/3$ and $t=5/6$; q_2,q_4 collide at $t=1/12$ and $t=1/4$ and $t=5/12$ and $t=7/12$ and $t=3/4$ and $t=11/12$.

3°. If q_2,q_3 collide at $t=0$ then q_2,q_3 collide at $t=0$ and $t=1/3$ and $t=2/3$; q_3,q_4 collide at $t=1/12$ and $t=5/12$ and $t=3/4$; q_4,q_1 collide at $t=1/6$ and $t=1/2$ and $t=5/6$; q_1,q_2 collide at $t=1/4$ and $t=7/12$ and $t=11/12$.

4°. If q_2,q_4 collide at $t=0$ then q_2,q_4 collide at $t=0$ and $t=1/6$ and $t=1/3$ and $t=1/2$ and $t=2/3$ and $t=5/6$; q_1,q_3 collide at $t=1/12$ and $t=1/4$ and $t=5/12$ and $t=7/12$ and $t=3/4$ and $t=11/12$.

By Lagrangian identity, we split the kinetic energy ([23,24,25]).

$$\sum_{i<j}|\dot q_i-\dot q_j|^2=\frac{1}{2}\sum_{i\neq j}|\dot q_i-\dot q_j|^2=\frac{1}{2}\sum_{i,j}(|\dot q_i|^2+\dot q_j|^2-2\langle\dot q_i,\dot q_j\rangle$$
$$=\sum_{i=1}^4|\dot q_i|^2\sum_j 1-\langle\sum_i\dot q_i,\sum_j\dot q_j\rangle=4\sum_{i=1}^4|\dot q_i|^2.$$

Thus

$$f(q)=\int_0^T(\frac{1}{2}\sum_{i=1}^4|\dot q_i|^2-V(q))dt$$
$$=\frac{1}{4}\sum_{1\leq i<j\leq 4}(\frac{1}{2}\int_0^1|\dot q_i-\dot q_j|^2dt+\int_0^1\frac{4}{|q_i-q_j|}dt).$$

We notice that the Lagrangian action for 4–body problems is the sum for the lagrangian actions for six two–body problems with the same weights. We want to use Lemma 2.3 to estimate the lower bound for the Lagrangian action of 4–body problems on collision generalized solutions. There are four cases for collisions, but from Gordon's Lemma, we notice that the more for collision times, the larger of Lagrangian action for the relative Keplerian problems for the collision. So we only consider these cases.

For case 1 we have

$$f(q) \geq \frac{3}{2}(2\pi)^{2/3}4^{-1/3}[((\frac{1}{12})^{\frac{1}{3}} + (\frac{1}{4})^{\frac{1}{3}}) \times 12 + 2] \approx 47.6229.$$

For case 2 we have

$$f(q) \geq \frac{3}{2}(2\pi)^{2/3}4^{-1/3}[(\frac{1}{6})^{\frac{1}{3}} \times 12 + 4] \approx 34.1184.$$

For case 3 we have

$$f(q) \geq \frac{3}{2}(2\pi)^{2/3}4^{-1/3}[(\frac{1}{3})^{\frac{1}{3}} \times 12 + 2] \approx 33.2061.$$

For case 4 we have

$$f(q) \geq \frac{3}{2}(2\pi)^{2/3}4^{-1/3}[(\frac{1}{6})^{\frac{1}{3}} \times 12 + 4] \approx 34.1184.$$

We can see that the infimum of the action functional on the collision set is larger than 33.2061.

Finally, we estimate the upper bound of $inf\{f(q), q \in \Lambda\}$. We choose three rose orbits as the test loops. Let $a > 0$ and

$$\begin{cases} q_1(t) = (a\sin 3\pi t \cos \pi t, a \sin 3\pi t \sin \pi t)^T, \\ q_2(t) = q_1(t + \frac{1}{4}), \\ q_3(t) = q_1(t + \frac{1}{2}), \\ q_4(t) = q_1(t + \frac{3}{4}). \end{cases}$$

Let $a = 0.3$, then we get

$$f(q) \approx 22.4945 < 33.2061.$$

In the estimate of the upper bound of $inf\{f(q), q \in \Lambda\}$ and the infimum of the action functional on the collision set,we use Mathematica and the accuracy is 0.0001.

$$f(q) \approx 20.1293 < 23.0852.$$

This proves the the minimizer of $f(q)$ on $\overline{\Lambda}$ is non-collision.

Proposition 3.1 together with Proposition 3.2 imply that the minimum of f on $\overline{\Lambda}$ is a non-collision solution to the four body problem (1.1). This proves Theorem 1.1.

References

[1] G. Arioli, F. Gazzola, S. Terracini: Minimization properties of Hill's orbits and applications to some N–body problems. *Ann. Inst. H.Poincare Anal. Non Lineaire.* 17,617–650(2000)

[2] A. Bahri, P. Rabinowitz: Periodic solutions of Hamiltonian systems of three body type. *Ann. IHP. nonlineaire*, 8, 561–649 (1991)

[3] U. Bessi, V. Coti Zelati: Symmetries and noncollision closed orbits for planar N-body-type problems. *Nonl. Anal.* 16, 587-598(1991)

[4] G. Butta , M. Giaquinta, S. Hildebrandt: One–dimensional variational problems. *Clarendon Press*: *Oxford*(1998)

[5] K-C Chen: Action minimizing orbits in the parallelogram four-body problem with equal masses. *Arch. Rational Mech. Anal.* 158, 293-318(2001)

[6] K-C Chen: Binary decompositions for planar N–body problems and symmetric periodic solutions. *Arch. Rational Mechanics Analysis.* 170, 247–276(2003)

[7] K-C Chen: Variational methods on periodic and quasi–periodic solutions for the N–body problems. *Ergodic Theory and Dynamical Systems.* 23, 1691–1715(2003)

[8] A. Chenciner, R. Montgomery: A remarkable periodic solution of the three body problem in the case of equal masses. *Annals of Math.* 152, 881–901(2000)

[9] A. Chenciner: Action minimizing solutions of the Newtonian n–body problem: From homology to symmetry. *ICM*. Vol.3, 279–294, Vol. 1, 641–643(2002)

[10] V. Coti Zelati: The periodic solutions of N-body type probliems. *Ann. IHP. Anal. nonlineaire.* 477-492(1990)

[11] M. Degiovanni, F. Giannoni: Dynamical systems with Newtonian type potentials, *Ann. Scuola Norm. Sup. Pisa.* 15, 467-494(1989)

[12] D. Ferrario, S. Terracini: On the existence of collisionless equivariant minimizers for the classical n–body problem. *Invention Math*. 155, 305–362(2004)

[13] W. Gordon: A minimizing property of Keplerian orbits. *Amer. J. Math.* 5, 961-971(1977)

[14] Y. Long, S. Q. Zhang: Geometric characterization for variational minimization solutions of the 3-body problem with fixed energy.*J.D.E.*160,422-438(2000)

[15] Y.M. Long, S. Q. Zhang: Goemetric characterizations for variational minimization solutions of the 3–body problems. *Acta Math. Sinica.* 16, 579–592(2000)

[16] C. Marchal: How the method of minimization of action avoids singularities. *Cel. Mech. and Dyn. Astronomy.* 83, 325–353(2002)

[17] C. Moore: Braids in classical gravity. *Physical Review Letters.* 70, 3675–3679(1993)

[18] R. Palais: The principle of symmetric criticality. *Comm. Math. Phy.* 69, 19-30(1979)

[19] C. Simo: Dynamical properties of the figure eight solution of the three-body problem,Contemp.*Math.292,AMS Providence,RI*,209-228(2002)

[20] C. Simo: New families of solutions in N–body problems. *Progress in Math.* 201, 101–115(2001)

[21] M. Struwe: Variational methods. *Springer–Verlag*(1990)

[22] A. Venturelli: Une caracterisation variationnelle des solutions de Lagrange du probleme plan des trois corps. *C.R.Acad. Sci. Paris.* 332, 641–644(2001)

[23] S. Q. Zhang: Periodic solutions for N-body problems. *June 1999;in Progress in Nonlinear Analysis ,edited by K.C.Chang and Y.M.Long.World Scientific.*423-443(2000)

[24] S.Q. Zhang, Q. Zhou: A minimizing property of Lagrangian solutions. *Acta Math. Sinica.* 17, 497–500(2001)

[25] S.Q. Zhang, Q. Zhou: Variational methods for the choreography solution to the three-body problem. *Science in China.* 45,594-597(2002)