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On the Existence of Three Solutions for a Dirichlet Boundary Value Problem in N-dimensional Case

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Abstract: In this note, we establish the existence of three weak solutions to the Dirichlet boundary value problem

$$\begin{cases} \Delta_p u + \lambda f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u$ =div$(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\Omega \subset R^N (N \geq 1)$ is non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $\lambda > 0$ and $f : R \to R$ is a continuous function. The result is based on a recent three critical points theorem.

Key words: Dirichlet problem; multiplicity results; critical point; three solutions
AMS subject classification: 35J20; 34A15.

1 Introduction

In this work, we study the boundary value problem

$$\begin{cases} \Delta_p u + \lambda f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u$ =div$(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\Omega \subset R^N (N \geq 1)$ is non-empty bounded open set with smooth boundary $\partial\Omega$, $p > N$, $\lambda > 0$ and $f : R \to R$ is a continuous function. Let us recall that a weak solution of problem (1) is any $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega (|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x))dx - \lambda \int_\Omega f(u(x))v(x)dx = 0, \quad \forall v \in W_0^{1,p}(\Omega).$$

Problem of the above type were widely studied in these latest years and we refer to [1-6] and the references therein for more details.

For instance, in [1], using variational methods, the authors ensure the existence of a sequence of arbitrarily small positive solutions for problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

when the Carathéodory function $f : \Omega \times R \to R$ has a suitable oscillating behaviour at zero, and in [4], using variational methods, the authors ensure the existence at least three weak solutions for the problem (2).

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Also, in [2] authors studied problem

$$\begin{cases} u'' + \lambda f(u) = 0, \\ u(0) = u(1) = 0, \end{cases} \tag{1.3}$$

by using a multiple fixed-point theorem to obtain three symmetric positive solutions under growth conditions on f.

In [3], the author proves multiplicity results for the problem (3) which for each $\lambda \in [0, +\infty[$, admits at least three solutions in $W_0^{1,2}([0,1])$ where $f : R \to R$ is continuous function.

Also, M. Ramaswamy and R. Shivaji recently in [5] established the existence of three positive solutions for classes of nondecreasing, p-sublinear functions f belonging to $C^1([0,\infty))$ for a p-Laplacian version of [2], i.e., the problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $p > 1$, $\lambda > 0$ is a parameter and Ω is a bounded domain in R^N; $N \geq 2$ with $\partial\Omega$ of class C^2 and connected.

In the present paper, under novel assumptions, we are interested in ensuring the existence of at least three weak solutions for the problem (1).

Our approach is based on a three critical points theorem proved in [8], recalled below for the reader's convenience (Theorem 1.1), and on technical lemma that allow us to apply it. Theorem 2.2 which is our main result, under novel assumptions ensures the existence of an open interval $\Lambda \subseteq [0, \infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1) admits at least three weak solutions whose norms in $W_0^{1,p}(\Omega)$ are less than q. As a consequence of Theorem 2.2, we obtain Corollary 2.3 and Theorem 2.4.

Corollary 2.3 ensures the existence of three weak solutions for the problem (1) when $f(t) \geq 0$ for each $t \in [-\beta, \max\{\beta, \alpha\}] (\alpha, \beta \in R)$.

Theorem 2.4 deals with the case $N = 1$, $p = 2$ and it ensures that, for any continuous function $f : R \to R$, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the problem (3) admits at least three solutions whose norms in $W_0^{1,2}([0,1])$ are less than q, as Example 2.5 shows.

The aim of the present paper is to extend the main result of [3] to the problem (1).

Finally, we here recall for the reader's convenience the three critical points theorem of [8], Proposition 3.1 of [7]:

Theorem 1.1. Let X be a separable and reflexive real Banach space; $\Phi : X \longrightarrow R$ a continuously $G\widehat{a}teaux$ differentiable and sequentially weakly lower semicontinuous functional whose $G\widehat{a}teaux$ derivative admits a continuous inverse on X^*; $\Psi : X \longrightarrow R$ a continuously $G\widehat{a}teaux$ differentiable functional whose $G\widehat{a}teaux$ derivative is compact. Assume that

$$\lim_{||u|| \to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty$$

for all $\lambda \in [0, +\infty[$, and that there exists a continuous concave function $h : [0, \infty[\to R$ such that

$$\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda\Psi(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda\Psi(u) + h(\lambda)).$$

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three solutions in X whose norms are less than q.

Proposition 1.2. Let X be a non-empty set and Φ, J two real function on X. Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that

$$\Phi(x_0) = J(x_0) = 0, \ \Phi(x_1) > r,$$

$$\sup_{x\in\Phi^{-1}(]-\infty,r])} J(x) < r\frac{J(x_1)}{\Phi(x_1)}.$$

Then, for each ρ satisfying

$$\sup_{x\in\Phi^{-1}(]-\infty,r])} J(x) < \rho < r\frac{J(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x)+\lambda(\rho-J(x))) < \inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x)+\lambda(\rho-J(x))).$$

2 Main results

Here and in the sequel, X will denote the Sobolev space $W_0^{1,p}(\Omega)$ with the norm

$$|| \, u \, || = \left(\int_\Omega |\nabla u(x)|^p dx\right)^{1/p},$$

and put

$$g(t) = \int_0^t f(\xi)d\xi$$

for each $t \in R$.

Now, fix $x^0 \in \Omega$ and pick r_1, r_2 with $o < r_1 < r_2$ such that

$$S(x^0, r_1) \subset S(x^0, r_2) \subseteq \Omega.$$

Put

$$k_1 = \frac{1}{r_2 - r_1}((r_2^N - r_1^N)\frac{\pi^{N/2}}{\Gamma(1+N/2)})^{1/p} c|\Omega|^{\frac{1}{N}-\frac{1}{p}} \tag{2.5}$$

and

$$k_2 = \frac{1}{r_2 - r_1}(\frac{r_2^N - r_1^N}{r_1^N})^{1/p} c|\Omega|^{\frac{1}{N}-\frac{1}{p}} \tag{2.6}$$

where Γ denotes the Gamma function, $c = c(N, p)$ is a positive constant and $|\Omega|$ is the measure of the set Ω.

Our main results fully depend on the following lemma:

Lemma 2.1. Assume that there exist two positive constants α and β with $k_1\alpha > \beta$ such that

(i) $g(t) \geq 0$ for each $t \in [0, \alpha]$,
(ii) $|\Omega|k_2^p \frac{\max_{t\in[-\beta,\beta]} g(t)}{\beta^p} < \frac{g(\alpha)}{\alpha^p}$,

where k_1 is given in (5) and k_2 by (6).
Then, there exist $r > 0$ and $w \in X$ such that $||w||^p > pr$ and

$$|\Omega| \max g(t) < pr\frac{\int_\Omega g(w(x))dx}{||w||^p}$$

where $t \in [-c|\Omega|^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}, c|\Omega|^{\frac{1}{N}-\frac{1}{p}} \sqrt[p]{pr}]$.

Proof: We put

$$w(x) = \begin{cases} 0 & , x \in \Omega \setminus S(x^0, r_2) \\ \frac{\alpha}{r_2 - r_1}[r_2 - \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}\,] & , x \in S(x^0, r_2) \setminus S(x^0, r_1) \\ \alpha & , x \in S(x^0, r_1) \end{cases}$$

and $r = \frac{\beta^p}{pc^p|\Omega|^{\frac{p}{N}-1}}$. It is easy to see that $w \in X$ and, in particular, one has

$$||w||^p = (r_2^N - r_1^N)\frac{\pi^{N/2}}{\Gamma(1+N/2)}(\frac{\alpha}{r_2 - r_1})^p.$$

Hence, taking into account that $k_1\alpha > \beta$, one has

$$pr < ||w||^p.$$

Since $0 \leq w(x) \leq \alpha$ for each $x \in \Omega$, condition (i) ensures that

$$\int_{\Omega \setminus S(x^0,r_2)} g(w(x))dx + \int_{S(x^0,r_2)\setminus S(x^0,r_1)} g(w(x))dx \geq 0.$$

Moreover, owing to our assumptions and since $\int_{S(x^0,r_1)} g(\alpha)dx = r_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)} g(\alpha)$, we have

$$\begin{aligned} |\Omega| \max g(t) < (\frac{\beta}{k_2\alpha})^p g(\alpha) &= \frac{\frac{\beta^p}{c^p|\Omega|^{\frac{p}{N}-1}}}{(r_2^N - r_1^N)\frac{\pi^{N/2}}{\Gamma(1+N/2)}(\frac{\alpha}{r_2-r_1})^p} r_1^N \frac{\pi^{N/2}}{\Gamma(1+N/2)} g(\alpha) \\ &= pr\frac{\int_{S(x^0,r_1)} g(\alpha)dx}{||w||^p} \\ &\leq pr\frac{\int_\Omega g(w(x))dx}{||w||^p}, \end{aligned}$$

where $t \in [-c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}, c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}]$.

So, the Proof is complete. □

Now, we state our main result which is used the argument of Theorems 2.1 and 2.2 in [4]:

Theorem 2.2. Assume that there exist four positive constants α, β, η and s with $k_1\alpha > \beta$ and $s < p$ such that

(i) $g(t) \geq 0$ for each $t \in [0, \alpha]$,
(ii) $|\Omega|k_2^p \frac{\max_{t\in[-\beta,\beta]} g(t)}{\beta^p} < \frac{g(\alpha)}{\alpha^p}$,
(iii) $g(t) \leq \eta(1 + |t|^s)$ for each $t \in R$,

where k_1 is given in (5) and k_2 by (6).
Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1) admits at least three solutions in X whose norms are less than q.

Proof: For each $u \in X$, we put

$$\Phi(u) = \frac{||u||^p}{p},$$

$$\Psi(u) = -\int_\Omega g(u(x))dx.$$

Of course, Φ is a continuously $G\widehat{a}teaux$ differentiable and sequentially weakly lower semi continuous functional whose $G\widehat{a}teaux$ derivative admits a continuous inverse on X^* and Ψ is a continuously $G\widehat{a}teaux$ differentiable functional whose $G\widehat{a}teaux$ derivative is compact. In particular, for each $u, v \in X$ one has

$$\Phi'(u)(v) = \int_\Omega (|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x))dx,$$

$$\Psi'(u)(v) = -\int_\Omega f(u(x))v(x)dx.$$

Hence, the weak solutions of (1) are exactly the solutions of the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0.$$

Thanks to (iii), for each $\lambda > 0$ one has that

$$\lim_{||u||\to+\infty}(\Phi(u) + \lambda\Psi(u)) = +\infty.$$

We claim that there exist $r > 0$ and $w \in X$ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < r\frac{(-\Psi(w))}{\Phi(w)}.$$

Now, taking into account that for every $u \in X$, one has

$$\sup_{x\in\Omega}|u(x)| \le c|\Omega|^{\frac{1}{N}-\frac{1}{p}}||u||$$

for each $u \in X$, it follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) = \sup_{||u||^p\le pr}\int_\Omega g(u(x))dx \le |\Omega|\max g(t)$$

where $t \in [-c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}, c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}]$.

Thanks to Lemma 2.1, there exist $r > 0$ and $w \in X$ such that

$$|\Omega|\max g(t) < pr\frac{\int_\Omega g(w(x))dx}{||w||^p},$$

where $t \in [-c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}, c|\Omega|^{\frac{1}{N}-\frac{1}{p}}\sqrt[p]{pr}]$.

So

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < r\frac{(-\Psi(w))}{\Phi(w)}.$$

Fix ρ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-\Psi(u)) < \rho < r\frac{(-\Psi(w))}{\Phi(w)}$$

and define $h(\lambda) = \lambda\rho$ for every $\lambda \ge 0$, from Proposition 1.2, with $x_0 = 0,\ x_1 = w,\ J = -\Psi$ we obtain

$$\sup_{\lambda\ge0}\inf_{u\in X}(\Phi(u) + \lambda\Psi(u) + \rho\lambda) < \inf_{u\in X}\sup_{\lambda\ge0}(\Phi(u) + \lambda\Psi(u) + \rho\lambda).$$

Now, our conclusion follows from Theorem 1.1. □

If $f(t) \ge 0$ for each $t \in [-\beta, \max\{\beta, \alpha\}]$. Then, by using of the Theorem 2.2, we have the following result:

Corollary 2.3. Assume that there exist four positive constants α, β, η and s with $k_1\alpha > \beta$ and $s < p$ such that

(i') $|\Omega|k_2^p\,\frac{g(\beta)}{\beta^p} < \frac{g(\alpha)}{\alpha^p}$,
(ii') $g(t) \le \eta(1 + |t|^s)$ for each $t \in R$,

where k_1 is given in (5) and k_2 by (6).
Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (1) admits at least three solutions in X whose norms are less than q.

We now want to point out a simple consequence of Theorem 2.2 in the case where $N = 1$ and $p = 2$.

For simplicity, we fix $\Omega =]0, 1[$ and consider a continuous functions $f : R \to R$. Moreover, put $g(t) = \int_0^t f(\xi)d\xi$ for all $t \in R$.

Taking into account that, in this situation, $c = \frac{1}{2}$, $k_1 = \sqrt{\frac{1}{2(r_2-r_1)}}$ and $k_2 = \frac{1}{2}\sqrt{\frac{1}{r_1(r_2-r_1)}}$, we have the following result:

Theorem 2.4. Assume that there exist four positive constants α, β, η and s with $\sqrt{\frac{1}{2(r_2-r_1)}}\ \alpha > \beta$ and $s < 2$ such that

(i'') $g(t) \geq 0$ for each $t \in [0, \alpha]$,
(ii'') $\frac{1}{4r_1(r_2-r_1)} \frac{\max_{t\in[-\beta,\beta]} g(t)}{\beta^2} < \frac{g(\alpha)}{\alpha^2}$,
(iii'') $g(t) \leq \eta(1 + |t|^s)$ for each $t \in R$.

Then, there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (3) admits at least three solutions in $W_0^{1,2}([0, 1])$ whose norms are less than q.

Example 2.5. Consider the problem

$$\begin{cases} u'' + \lambda(e^{1-u}u^6(7-u)) = 0, \\ u(0) = u(1) = 0. \end{cases} \tag{2.7}$$

Taking into account $c = \frac{1}{2}$, choosing $r_1 = \frac{1}{3}$, $r_2 = \frac{2}{3}$ and $f(u) = e^u u^2(3 + u)$ for each $u \in R$, so that $k_1 = \sqrt{\frac{3}{2}}$ and $k_2 = \frac{3}{2}$, all the assumptions of Theorem 2.4, are satisfied by choosing, for instance $\alpha = 2,\ \beta = 1,\ s = 1$, η sufficiently large. So there exists an open interval $\Lambda \subseteq [0, +\infty[$ and a positive real number q such that, for each $\lambda \in \Lambda$, problem (7) admits at least three solutions in $W_0^{1,2}([0, 1])$ whose norms are less than q.

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