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A Modified F-expansion Method for Solving Breaking Soliton Equation

Guoliang Cai [1], Qingchao Wang[2,*], Juanjuan Huang[3,+]
[1,2,3]Faculty of Science, Jiangsu University,
Jiangsu, Zhenjiang 212013, P.R. China
*Information College, Zhejiang Ocean University,
Zhoushan, Zhejiang, 316004, P.R.China
+Department of Mathematics, Yancheng Teachers' College,
Yancheng Jiangsu 224002, P.R. China
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Abstract. In this paper, a modified F-expansion method is proposed by taking full advantages of F-expansion method and Riccati equation in seeking exact solutions of non-linear partial differential equations. By using the method, rich families of exact solutions of Breaking Soliton Equation can be obtained, including soliton-like solutions, trigonometric function solutions and rational solutions. The method can be applied to solving massive non-linear partial differential equations(group), as well as helps us to find new exact solutions. Furthermore, with the aid of computer symbolic systems (Mathematica or Maple), the method can be conveniently operated. Finally, some figures of partial solutions are provided for direct-viewing analysis.

Keywords: modified F-expansion method; breaking soliton equation; Riccati equation; exact solution

1 Introduction

Numbers of mathematical models can be described by Nonlinear Partial Differential Equation(NLPDE), especially some basic equations in physics and mechanics. As a result, the research on exact solutions to NLPDE is becoming more and more important, such as the famous Inverse scattering method, Backlund transformation, Darboux transformation, Hirota bilinear method, Painleve method and so on[1-5]. In recent years, direct search for exact solutions to NLPDEs has become more and more attractive partly due to the availability of computer symbolic systems like Maple or Mathematica which allows us to perform some complicated and tedious algebraic calculation on computer, and helps us to find new exact solutions to PDEs, such as Homogeneous balance method[6], tanh-function method [7], sine-cosine method[8], Jacobi elliptic functions method[9], F-expansion[10,11] and so on.

In this paper,we put forward a modified F-expansion method by taking full advantages of F-expansion method and Riccati equation in seeking exact solutions of nonlinear PDEs. Before introducing the modified F-expansion method, we simply describe the F-expansion method as follows:

For the given NLPDE, given two variables x, t

$$P(u, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0$$

We seek its travelling wave solution in the formal solution

$$u(\xi) = a_0 + \sum_{i=-N}^{N} a_i F^i(\xi) \qquad (a_N \neq 0)$$

[1]**Corresponding author**. *E-mail address*: glcai@ujs.edu.cn

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With $F(\xi)$ satisfying the non-linear ODE

$$F'^2(\xi) = PF^4(\xi) + QF^2(\xi) + R$$

where $' = \frac{d}{d\xi}$, P, Q, R are constants, which is more powerful than Jacobi elliptic functions method. But the method can just be well used to solve the NLPDEs whose odd- and even-order derivatives terms do not coexist[11]. In order to overcome this disadvantage, we substitute Riccati equation $(F'(\xi) = A + BF(\xi) + CF^2(\xi))$ into the ODE$(F'^2(\xi) = PF^4(\xi) + QF^2(\xi) + R)$. In what follows we introduce the modified F-expansion method and apply it to Breaking Soliton Equation,

$$u_{xt} - 4u_x u_{xy} - 2u_y u_{xx} + u_{xxxy} = 0 \tag{1}$$

Eq.(1) could describe the (2+ 1)-dimensional interaction of Riemann wave propagation along the y-axis with long-wave propagation along the x-axis[12]. The equation has been researched by many scientists[13-15], and many exact solutions have been derived. Here we use our modified F-expansion method to solve it and get a series of its soliton-like solutions, trigonometric function solutions and rational solutions. In fact, we can apply the method to amounts of NLPDEs and obtain many new exact solutions.

2 Summary of our method

Consider a given NLPDE with independent variables $x = (x_1, x_2, ..., x_l, t)$ and dependent variable u

$$P(u, u_t, u_{x_1}, u_{tt}, ...) = 0, \tag{2}$$

Generally speaking, the left-hand side of Eq.(2) is a polynomial in u and its various partial derivatives. The main points of the modified F-expansion method for solving Eq.(2) are as follows:

Firstly, seek travelling wave solutions to Eq.(2) by taking

$$u(x_1, x_2, ..., x_l, t) = u(\xi), \qquad \xi = k_1(x_1 + k_2x_2 + ... + k_lx_l + \omega t) \tag{3}$$

where $k_1, k_2, \ldots, k_l, \omega$ are constants to be determined. Inserting (3) into Eq.(2) yields an ODE for $u(\xi)$

$$P(u, u', u'', ...) = 0 \tag{4}$$

Secondly, suppose that $u(\xi)$ can be expressed as

$$u(\xi) = a_0 + \sum_{i=-N}^{N} a_i F^i(\xi) \qquad (a_N \neq 0) \tag{5}$$

where a_0, $a_i$are constants to be determined. $F(\xi)$satisfies Riccati equation

$$F'(\xi) = A + BF(\xi) + CF^2(\xi) \tag{6}$$

where A, B, C are constants to be determined. Integer N can be determined by considering the homogeneous balance between the governing nonlinear term(s) and highest order derivatives of $u(\xi)$ in Eq.(4). And

1. when $N = \frac{p}{q}$ is fraction, let $u(\xi) = v^{\frac{p}{q}}(\xi)$

2. when N is negative integer, let $u(\xi) = v^N(\xi)$

we change Eq.(4) into another ODE for $v(\xi)$, whose balancing number will be a positive integer.

Thirdly, substitute (5) into Eq.(4), and using (6), and then the left-hand side of Eq.(4) can be converted into a finite series in $F^p(\xi)$ (p=-N,...,−1,0,1,...,N). Equating each coefficient of $F^p(\xi)$ to zero yields a system of algebraic equations for a_i $(i = -N, ..., -1, 0, 1, ..., N)$, $k_\lambda(\lambda = 1, ..., l)$, ω.

Fourthly, solve the system of algebraic equations, probably with the aid of *Mathematica* or *Maple*. a_i, k_λ, ωcan be expressed by A, B, C (or the coefficients of ODE(4)). Substituting these results into (5), we can obtain the general form of travelling wave solutions to Eq.(4).

Fifthly, with the aid of Appendix, from the general form of travelling wave solutions,we can give a series of soliton-like solutions, trigonometric function solutions and rational solutions to Eq.(2).

3 Exact solutions to breaking soliton equation

In this section, we will make use of the modified F-expansion method and symbolic computation to find the exact solutions to Breaking Soliton Equation.

(i) We assume that Eq.(1) has travelling wave solution in the form

$$u(x,y,t)=u(\xi),\qquad \xi=k(x+ly+\omega t)\quad (k\neq 0) \tag{7}$$

Substituting (7) into (1)we have

$$k^2\omega u''-6k^3lu'u''+k^4lu^{(4)}=0 \tag{8}$$

or

$$\omega u''-6klu'u''+k^2lu^{(4)}=0 \tag{9}$$

(ii) Considering the homogeneous balance between $u'u''$and $u^{(4)}$ in (9), we suppose that the solution to ODE(9) can be expressed by

$$u(\xi)=a_0+a_{-1}F(\xi)^{-1}+a_1F(\xi) \tag{10}$$

where a_0, a_{-1}, $a_1$are constants to be determined. Substituting (10) into Eq.(9), and using (6), the left-hand side of Eq.(9) can be converted into a finite series in $F^p(\xi)$(p=-5,. . . ,-1,0,1,. . . ,5). Equating each coefficient of $F^p(\xi)$to zero yields a system of algebraic equations for a_{-1}, a_0, a_1, l, ω.

$$F^5: 24C^4k^4la_1-12C^3k^3la_1^2=0 \tag{11.1}$$

$$F^4: 60BC^3k^4la_1-30BC^2k^3la_1^2=0 \tag{11.2}$$

$$F^3: \begin{array}{l}50B^2C^2k^4la_1+40AC^3k^4la_1+2C^2k^2\omega a_1+12C^3k^3la_{-1}a_1-\\ 24B^2Ck^3la_1^2-24AC^2k^3la_1^2=0\end{array} \tag{11.3}$$

$$F^2: \begin{array}{l}15B^3Ck^4la_1+60ABC^2k^4la_1+3BCk^2\omega a_1+24BC^2k^3la_{-1}a_1-\\ 6B^3k^3la_1^2-36ABCk^3la_1^2=0\end{array} \tag{11.4}$$

$$F^1: \begin{array}{l}B^4k^4la_1+22AB^2Ck^4la_1+16A^2C^2k^4la_1+B^2k^2\omega a_1+2ACk^2\omega a_1+\\ 12B^2Ck^3la_{-1}a_1+12AC^2k^3la_{-1}a_1-12AB^2k^3la_1^2-12A^2Ck^3la_1^2=0\end{array} \tag{11.5}$$

$$F^0 \begin{array}{l}B^3Ck^4la_{-1}+8ABC^2k^4la_{-1}+BCk^2\omega a_{-1}+6BC^2k^3la_{-1}^2+AB^3k^4la_1+\\ 8A^2BCk^4la_1+ABk^2\omega a_1-6A^2Bk^3la_1^2=0\end{array} \tag{11.6}$$

$$F^{-1}: \begin{array}{l}B^4k^4la_{-1}+22AB^2Ck^4la_{-1}+16A^2C^2k^4la_{-1}+B^2k^2\omega a_{-1}+2ACk^2\omega a_{-1}+\\ 12B^2Ck^3la_{-1}^2+12AC^2k^3la_{-1}^2-12AB^2k^3la_{-1}a_1-12A^2Ck^3la_{-1}a_1=0\end{array} \tag{11.7}$$

$$F^{-2}: \begin{array}{l}15AB^3k^4la_{-1}+60A^2BCk^4la_{-1}+3ABk^2\omega a_{-1}+6B^3k^3la_{-1}^2+36ABCk^3la_{-1}^2-\\ 24A^2Bk^3la_{-1}a_1=0\end{array} \tag{11.8}$$

$$F^{-3}: \begin{array}{l}50A^2B^2k^4la_{-1}+40A^3Ck^4la_{-1}+2A^2k^2\omega a_{-1}+24AB^2k^3la_{-1}^2+24A^2Ck^3la_{-1}^2-\\ 12A^3k^3la_{-1}a_1=0\end{array} \tag{11.9}$$

$$F^{-4}: 60A^3Bk^4la_{-1}+30A^2Bk^3la_{-1}^2=0 \tag{11.10}$$

$$F^{-5}: 24A^4k^4la_{-1}+12A^3k^3la_{-1}^2=0 \tag{11.11}$$

(iii) Solving the algebraic equation(11) using *Mathematica*, we have the following solutions: a_{-1}, a_0, a_1, l, ω

*Case*1: when A=0, we have

$$a_0=a_0,\qquad a_{-1}=0,\qquad a_1=2Ck,\qquad l=l,\qquad \omega=-B^2k^2l \tag{12}$$

*Case*2: when$B$=0, we have

$$a_0=a_0,\qquad a_{-1}=0,\qquad a_1=2Ck,\qquad l=l,\qquad \omega=4ACk^2l \tag{13}$$

$$a_0=a_0,\qquad a_{-1}=-2Ak,\qquad a_1=2Ck,\qquad l=l,\qquad \omega=16ACk^2l \tag{14}$$

*Case*3: when$A = B$=0, we have

$$a_0 = a_0, \qquad a_{-1} = 0, \qquad a_1 = 2Ck, \qquad l = l, \qquad \omega = 0 \tag{15}$$

$$a_0 = a_0, \qquad a_{-1} = -\frac{\omega}{6Ckl}, \qquad a_1 = 2Ck, \qquad l = l, \qquad \omega = \omega \tag{16}$$

Substituting these solutions into (10), from Appendix, we can obtain many soliton-like solutions, trigonometric function solutions and rational solutions to Eq.(1)(where we left the same type solutions out):

3.1 The soliton-like solutions to Eq.(1)

(1) When A=0, B=1, C= -1, from Appendix, $F(\xi) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}\xi)$. By *case*1, we have

$$u_1 = a_0 - k - k\tanh[\frac{1}{2}k(x + ly - k^2lt)]$$

(2) When A=0, B= -1, C=1, from Appendix, $F(\xi) = \frac{1}{2} - \frac{1}{2}\coth(\frac{1}{2}\xi)$. By *case*1, we have

$$u_2 = a_0 + k - k\coth[\frac{1}{2}k(x + ly - k^2lt)]$$

(3) When $A = \frac{1}{2}$, B=0, $C = -\frac{1}{2}$, from Appendix, $F(\xi) = \coth\xi \pm \text{csch}\xi$ or $\tanh\xi \pm i\text{sech}\xi$. By *case*2, we have

$$u_3 = a_0 - k\coth[k(x + ly - k^2lt)] \mp k\csc h[k(x + ly - k^2lt)]$$
$$u_4 = a_0 - k\tanh[k(x + ly - k^2lt)] \mp ik\sec h[k(x + ly - k^2lt)]$$

For direct-viewing analysis,we provide the figures of u_3,where we choose a_0=0, $k = l$=1, t=0.

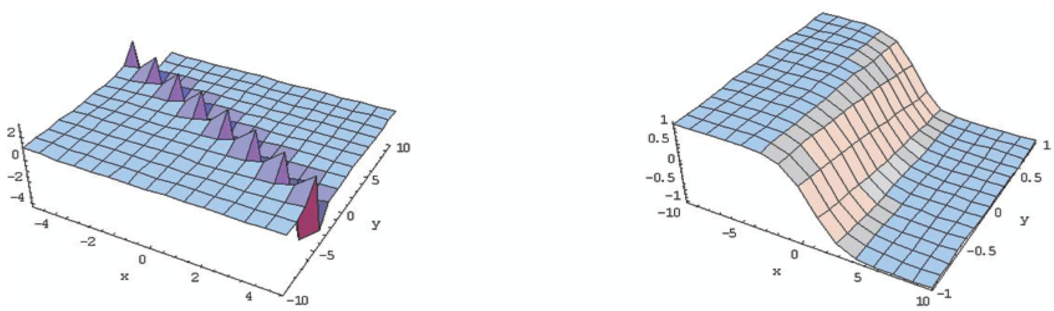

Figure 1: Graphics of soliton-like solution u_3 are shown at "-" and "+", respectively

(4) When $A = 1$, B=0, $C = -1$, from Appendix , $F(\xi) = \tanh\xi$ or $\coth\xi$. By *case*2, we have

$$u_5 = a_0 - 2k\tanh[k(x + ly - 16k^2lt)] - 2k\coth[k(x + ly - 16k^2lt)]$$

3.2 The trigonometric function solutions to Eq.(1)

(1) When $A = C = \frac{1}{2}$, B=0, from Appendix , $F(\xi)$=secξ+tanξ or cscξ-cotξ. By *case*2, we have

$$u_6 = a_0 + k\sec[k(x + ly + k^2lt)] + k\tan[k(x + ly + k^2lt)]$$

$$u_7 = a_0 + k\csc[k(x + ly + k^2lt)] - k\cot[k(x + ly + k^2lt)]$$

$$u_8 = a_0 + 2k\tan[k(x + ly + 4k^2lt)]$$

$$u_9 = a_0 - 2k\cot[k(x + ly + 4k^2lt)]$$

(2) When $A = C = -\frac{1}{2}$, B=0, from Appendix , $F(\xi)$=secξ-tanξ or csc ξ+cotξ. By *case*2, we have

$$u_{10} = a_0 - k\sec[k(x + ly + k^2lt)] + k\tan[k(x + ly + k^2lt)]$$

$$u_{11} = a_0 - k\csc[k(x + ly + k^2lt)] - k\cot[k(x + ly + k^2lt)]$$

(3) When $A = C = 1$, B=0, from Appendix , $F(\xi)$=tanξ. By *case*2, we have

$$u_{12} = a_0 + 2k\tan[k(x + ly + 16k^2lt)] - 2k\cot[k(x + ly + 16k^2lt)]$$

where a_0, k, l are arbitrary constants in section 3.1, 3.2.

3.3 The rational solutions to Eq.(1)

When $A = B$=0, $C \neq$0, from Appendix , $F(\xi) = -\frac{1}{C\xi+\lambda}$. By *case*3, we have

$$u_{13} = a_0 - \frac{2Ck}{Ck(x + ly) + \lambda}$$

$$u_{14} = a_0 + \frac{\omega}{6Ckl}[Ck(x + ly + \omega t) + \lambda] - \frac{2Ck}{Ck(x + ly + \omega t) + \lambda}$$

where a_0, C, k, l, ω are arbitrary constants and $C \neq$0.

As we know, the exact solutions u_3, u_4, u_5, u_{12}, u_{14} are firstly derived by us (the similar type of the rest solutions appear in[14], [15]). Next, we will provide the figures of u_5, u_6, u_7, u_{12} for direct-viewing analysis. We choose a_0=0, $k = l$=1, t=0.

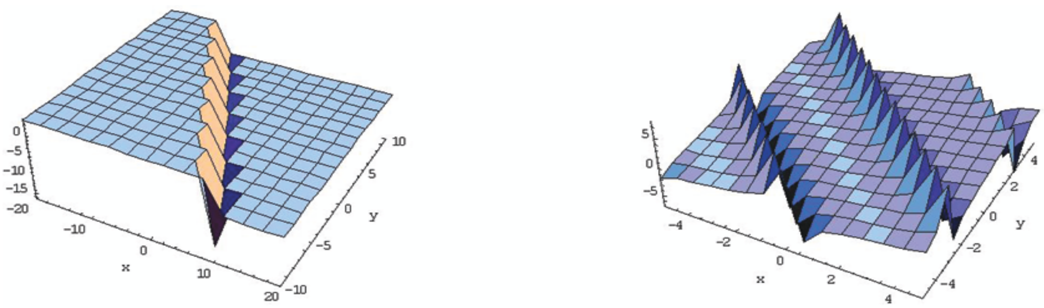

Figure 2: (a) Graphic of the soliton-like solution u_5 (b) Graphic of the periodic solution u_6

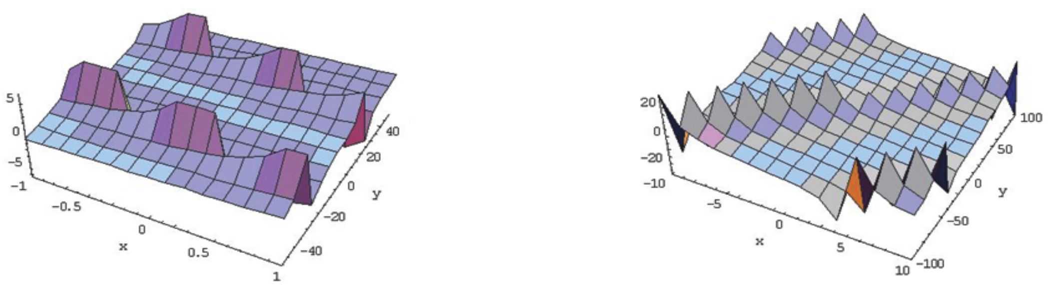

Figure 3: (a) Graphic of periodic solution u_7 (b) Graphic of periodic solution u_{12}

4 Conclusions

In this paper, by using our modified F-expansion method, we have considered the Breaking Soliton Equation and derived many types of exact solutions to it, including soliton-like solutions, trigonometric function solutions and rational solutions, some of which have not appeared in those known literatures. They should be meaningful to explain some physics phenomena. We can also see that the method overcome some disadvantages of F-expansion method and can be applied to more NLPDEs. Moreover, with the aid of computer symbolic systems (*Mathematica* or *Maple*), the method can be conveniently operated.

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Appendix

Relations between values of the parameters (A, B, C) and the corresponding solutions $F(\xi)$ in Riccati equation

$$F'(\xi) = A + BF(\xi) + CF^2(\xi).$$

A	B	C	F
0	1	-1	$\frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}\xi)$
0	-1	1	$\frac{1}{2} - \frac{1}{2}\coth(\frac{1}{2}\xi)$
$\frac{1}{2}$	0	$-\frac{1}{2}$	$\coth\xi \pm \text{csch}\xi, \tanh\xi \pm i\text{sech}\xi$
1	0	-1	$\tanh\xi, \coth\xi$
$\frac{1}{2}$	0	$\frac{1}{2}$	$\sec\xi + \tan\xi, \csc\xi - \cot\xi$
$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\sec\xi - \tan\xi, \csc\xi + \cot\xi$
1(-1)	0	1(-1)	$\tan\xi(\cot\xi)$
0	0	$\neq 0$	$-\frac{1}{C\xi+\lambda}$ (λ is arbitrary constant)
arbitrary constants	0	0	$A\xi$
arbitrary constants	$\neq 0$	0	$\frac{\exp(B\xi)-A}{B}$

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