ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.6(2008) No.1,pp.11-20

Modified Reductive Perturbation Method as Applied to Long Water-Waves:The Korteweg-de Vries Hierarchy

Hilmi Demiray *
Department of Mathematics,Faculty of Arts and Sciences
Isik University, Sile 34980-Istanbul-Turkey

(Received 6 July 2007, accepted 14 June 2008)

Abstract: In this work, we extended the application of "the modified reductive perturbation method" to long water waves and obtained the governing equations of KdV hierarchy. Seeking a localized travelling wave solutions to these evolution equations we determined the scale parameter c_1 so as to remove the possible secularities that might occur. To indicate the power and elegance of the present method, we studied the same problem by use of the " dressed solitary wave method " and obtained exactly the same result. The present method is seen to be fairly simple as compared to the renormalization method of Kodama and Taniuti [4] and the multiple scale expansion method of Kraenkel et al [6].

Keywords:modified reductive perturbation method; water waves; Korteweg-deVries hierarchy

1 Introduction

In collisionless cold plasma, in fluid-filled elastic tubes and in shallow-water waves, due to nonlinearity of the governing equations, for weakly dispersive case one obtains the Korteweg-de Vries (KdV) equation for the lowest order term in the perturbation expansion, the solution of which may be described by solitons (Davidson [1]). To study the higher order terms in the perturbation expansion, the reductive perturbation method has been introduced by use of the stretched time and space variables (Taniuti [2]). However, in such an approach some secular terms appear which can be eliminated by introducing some slow scale variables (Sugimoto and Kakutani [3]) or by a renormalization procedure of the velocity of the KdV soliton (Kodama and Taniuti [4]). Nevertheless, this approach remains somewhat artificial, since there is no reasonable connection between the smallness parameters of the stretched variables and the one used in the perturbation expansion of the field variables. The choice of the former parameter is based on the linear wave analysis of the concerned problem and the wave number or the frequency is taken as the perturbation parameter (Washimi and Taniuti [5]). On the other hand, at the lowest order, the amplitude and the width of the wave are expressed in terms of the unknown perturbed velocity, which is also used as the smallness parameter. This causes some ambiguity over the correction terms. Another attempt to remove such secularities is made by Kraenkel et al [6] for long water waves by use of the multiple time scale expansion but could not obtain explicitly the correction terms to the wave speed.

In order to remove these uncertainities, Malfliet and Wieers [7] presented a dressed solitary wave approach, which is based on the assumption that the field variables admit localized travelling wave solution. Then, for the longwave limit, they expanded the field variables and the wave speed into a power series of the wave number, which is assumed to be the only smallness parameter, and obtained the explicit solution for various order terms in the expansion. However, this approach can only be used when one studies progressive wave solution to the original nonlinear equations and it does not give any idea about the form of evolution equations governing the various order terms in the perturbation expansion. In our previous paper [8], we

* *E-mail address*: demiray@isikun.edu.tr

IJNS.2008.08.15/157

have presented a method so called "the modified reductive perturbation method" to examine the contributions of higher order terms in the perturbation expansion and applied it to weakly dispersive ion-acoustic plasma waves and solitary waves in a fluid filled elastic tube [9]. In these works, we have shown that the lowest order term in the perturbation expansion is governed by the nonlinear Korteweg-de Vries equation, whereas the higher order terms in the expansion are governed by the degenerate Korteweg-de Vries equation with non-homogeneous term. By employing the hyperbolic tangent method a progressive wave type of solution was sought and the possible secularities were removed by selecting the scaling parameter in a special way. The basic idea in this method was the inclusion of higher order dispersive effects through the introduction of the scaling parameter c, to balance the higher order nonlinearities with dispersion. The negligence of higher order dispersive effects in the classical reductive perturbation method leads to the imbalance between the nonlinearity and the dispersion, which resulted in some secular terms in the solution of evolution equations. As a matter of fact, the renormalization method presented by Kodama and Taniuti [4] is different but rather involved formulation of the same idea. For further discussion of theses methods the reader is referred to the references [10-13].

In the present work, we extended the application of "the modified reductive perturbation method" to long water-waves and obtained the equations of KdV hierarchy. Seeking a localized progressive wave solutions to these equations we determined the scale parameter c_1 so as to remove the possible secularities that might occur. To indicate the power and elegance of the present method, we studied the same problem by use of the " dressed solitary wave method " and obtained exactly the same result. The present method is seen to be fairly simple as compared to the renormalization method of Kodama and Taniuti [4] and the multiple scale expansion method of Kraenkel et al [6].

2 Modified reductive perturbation formalism for water waves

We consider a two-dimensional incompressible inviscid fluid in a constant gravitational field g. The space coordinates are denoted by (x, z) and the corresponding velocity components by (u, w). The gravitational acceleration is in negative z direction. The equations describing such a fluid are:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, (\text{incompressibility}) \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g, \tag{3}$$

where ρ is the mass density and p is the fluid pressure function. Assuming that the flow is irrotational, the velocity vector can be derived from a scaler potential $\hat{\phi}(x, z, t)$ as

$$u=\frac{\partial \hat{\phi}}{\partial x}, \quad w=\frac{\partial \hat{\phi}}{\partial z}. \tag{4}$$

Then, the incompressibility condition reduces to

$$\frac{\partial^2 \hat{\phi}}{\partial x^2}+\frac{\partial^2 \hat{\phi}}{\partial z^2}=0, \tag{5}$$

and the Euler equations become

$$\frac{p-p_0}{\rho}=-\frac{\partial \hat{\phi}}{\partial t}-\frac{1}{2}[(\frac{\partial \hat{\phi}}{\partial x})^2+(\frac{\partial \hat{\phi}}{\partial z})^2]-gz, \tag{6}$$

where p_0 is an integration constant.

We consider the case of a fluid of height h, bounded above by a steady atmosphere with pressure p_0. Let the upper surface be described by $z=\hat{\psi}(x, t)$. The kinematic boundary condition on this surface can be expressed as:

$$\frac{\partial \hat{\phi}}{\partial z}=\frac{\partial \hat{\psi}}{\partial t}+\frac{\partial \hat{\phi}}{\partial x}\frac{\partial \hat{\psi}}{\partial x}, \quad \text{on} \quad z=\hat{\psi}(x, t). \tag{7}$$

From equation (6), the dynamical boundary condition on this surface reads

$$\frac{\partial\hat{\phi}}{\partial t}+\frac{1}{2}[(\frac{\partial\hat{\phi}}{\partial x})^2+(\frac{\partial\hat{\phi}}{\partial z})^2]+g\hat{\psi}=0,\quad \text{on}\quad z=\hat{\psi}(x,t). \tag{8}$$

Finally, the lower boundary is supposed to be rigid. Therefore, at $z=-h$, the normal component of the velocity must vanish, $i.e.$,

$$\frac{\partial\hat{\phi}}{\partial z}=0\quad \text{at}\quad z=-h. \tag{9}$$

Now, we shall consider the longwave in shallow-water approximation to the above equations by applying the modified reductive perturbation method developed by us (Demiray [8]). According to this method, we introduce the following coordinate stretching

$$\xi=\epsilon^{1/2}(x-c_0t),\quad \tau=\epsilon^{3/2}ct, \tag{10}$$

where ϵ is a small parameter characterizing the smallness of certain physical entities, c_0 and c are two constants to be determined from the solution. For our future purposes we introduce new variable as

$$\hat{\phi}=\epsilon^{1/2}\phi,\quad \hat{\psi}=\epsilon\psi. \tag{11}$$

Introducing (10) and (11) into equations (5)-(9) we obtain

$$\epsilon\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial z^2}=0, \tag{12}$$

$$\frac{\partial\phi}{\partial z}=\epsilon^2c\frac{\partial\psi}{\partial\tau}-\epsilon c_0\frac{\partial\psi}{\partial\xi}+\epsilon^2\frac{\partial\phi}{\partial\xi}\frac{\partial\psi}{\partial\xi}\quad \text{at}\quad z=\epsilon\psi, \tag{13}$$

$$\epsilon c\frac{\partial\phi}{\partial\tau}-c_0\frac{\partial\phi}{\partial\xi}+\frac{1}{2}[\epsilon(\frac{\partial\phi}{\partial\xi})^2+(\frac{\partial\phi}{\partial z})^2]+g\psi=0,\ \text{at}\quad z=\epsilon\psi, \tag{14}$$

and

$$\frac{\partial\phi}{\partial z}=0,\quad \text{at}\quad z=-h. \tag{15}$$

Now, we expand the functions ϕ and ψ, and the constant c into a suitable power series in the parameter ϵ :

$$\phi=\phi_0+\epsilon\phi_1+\epsilon^2\phi_2+\epsilon^3\phi_3+....,\quad \psi=\psi_0+\epsilon\psi_1+\epsilon^2\psi_2+\epsilon^3\psi_3+....,$$
$$c=1+\epsilon c_1+\epsilon^2c_2+\epsilon^3c_3+...., \tag{16}$$

Introducing the expansions (16) into equations (12)-(15) the following set of equations are obtained:

$O(1)$ *equations*

$$\frac{\partial^2\phi_0}{\partial z^2}=0, \tag{17}$$

and the boundary conditions

$$\frac{\partial\phi_0}{\partial z}=0,\quad \text{at}\quad z=-h. \tag{18}$$

$$\frac{\partial\phi_0}{\partial z}|_{z=0}=0,\quad -c_0\frac{\partial\phi_0}{\partial\xi}|_{z=0}+g\psi_0=0. \tag{19}$$

$O(\epsilon)$ *equations*

$$\frac{\partial^2\phi_1}{\partial z^2}+\frac{\partial^2\phi_0}{\partial\xi^2}=0, \tag{20}$$

and the boundary conditions

$$\frac{\partial\phi_1}{\partial z}=0\quad \text{at}\quad z=-h,$$
$$\frac{\partial\phi_1}{\partial z}|_{z=0}+c_0\frac{\partial\psi_0}{\partial\xi}=0,\quad [\frac{\partial\phi_0}{\partial\tau}-c_0\frac{\partial\phi_1}{\partial\xi}+\frac{1}{2}(\frac{\partial\phi_0}{\partial\xi})^2]|_{z=0}+g\psi_1=0. \tag{21}$$

$O(\epsilon^2)$ equations

$$\frac{\partial^2\phi_2}{\partial z^2}+\frac{\partial^2\phi_1}{\partial\xi^2}=0, \tag{22}$$

and the boundary conditions

$$\frac{\partial\phi_2}{\partial z}=0 \quad \text{at} \quad z=-h,$$

$$[\frac{\partial\phi_2}{\partial z}+\psi_0\frac{\partial^2\phi_1}{\partial z^2}]|_{z=0}=\frac{\partial\psi_0}{\partial\tau}-c_0\frac{\partial\psi_1}{\partial\xi}+\frac{\partial\phi_0}{\partial\xi}\frac{\partial\psi_0}{\partial\xi}|_{z=0},$$

$$[c_1\frac{\partial\phi_0}{\partial\tau}+\frac{\partial\phi_1}{\partial\tau}-c_0(\frac{\partial\phi_2}{\partial\xi}+\psi_0\frac{\partial^2\phi_1}{\partial z\partial\xi})+\frac{\partial\phi_0}{\partial\xi}\frac{\partial\phi_1}{\partial\xi}+\frac{1}{2}(\frac{\partial\phi_1}{\partial z})^2]|_{z=0}+g\psi_2=0. \tag{23}$$

$O(\epsilon^3)$ equations

$$\frac{\partial^2\phi_3}{\partial z^2}+\frac{\partial^2\phi_2}{\partial\xi^2}=0, \tag{24}$$

and the boundary conditions

$$\frac{\partial\phi_3}{\partial z}=0, \quad \text{at} \quad z=-h,$$

$$[\frac{\partial\phi_3}{\partial z}+\psi_0\frac{\partial^2\phi_2}{\partial z^2}+\psi_1\frac{\partial^2\phi_1}{\partial z^2}-\frac{\partial\phi_1}{\partial\xi}\frac{\partial\psi_0}{\partial\xi}-\frac{\partial\phi_0}{\partial\xi}\frac{\partial\psi_1}{\partial\xi}]|_{z=0}=c_1\frac{\partial\psi_0}{\partial\tau}+\frac{\partial\psi_1}{\partial\tau}-c_0\frac{\partial\psi_2}{\partial\xi},$$

$$[c_2\frac{\partial\phi_0}{\partial\tau}+c_1\frac{\partial\phi_1}{\partial\tau}+\frac{\partial\phi_2}{\partial\tau}+\psi_0\frac{\partial^2\phi_1}{\partial z\partial\tau}-c_0(\frac{\partial\phi_3}{\partial\xi}+\psi_0\frac{\partial^2\phi_2}{\partial z\partial\xi}+\frac{1}{2}\psi_0^2\frac{\partial^3\phi_1}{\partial z^2\partial\xi}+\psi_1\frac{\partial^2\phi_1}{\partial z\partial\xi})$$

$$\frac{1}{2}(\frac{\partial\phi_1}{\partial\xi})^2+\frac{\partial\phi_0}{\partial\xi}(\frac{\partial\phi_2}{\partial\xi}+\psi_0\frac{\partial^2\phi_1}{\partial z\partial\xi})+\frac{\partial\phi_1}{\partial z}(\frac{\partial\phi_2}{\partial z}+\psi_0\frac{\partial^2\phi_1}{\partial z^2}]|_{z=0}+g\psi_3=0. \tag{25}$$

2.1 Solution of the field equations

From the solution of the sets (18) and (19) we have

$$\phi_0=F(\xi,\tau), \quad \psi_0=\frac{c_0}{g}\frac{\partial F}{\partial\xi}, \tag{26}$$

where $F(\xi,\tau)$ is an unknown function of its argument whose governing equation will be obtained later.

Similarly, from the solution of the equations (20) and (21) one obtains

$$\phi_1=-\frac{1}{2}\frac{\partial^2F}{\partial\xi^2}(z^2+2hz)+G(\xi,\tau), \quad \psi_1=\frac{c_0}{g}\frac{\partial G}{\partial\xi}-\frac{1}{g}[\frac{\partial F}{\partial\tau}+\frac{1}{2}(\frac{\partial F}{\partial\xi})^2], \quad c_0=(gh)^{1/2}, \tag{27}$$

where $G(\xi,\tau)$ is another unknown function whose governing equation will be obtained from the higher order expansions.

The solution of $O(\epsilon^2)$ equations, (22) and (23), yields the following results

$$\phi_2=\frac{1}{24}\frac{\partial^4F}{\partial\xi^4}(z^4+4hz^3-8h^3z)-\frac{1}{2}\frac{\partial^2G}{\partial\xi^2}(z^2+2hz)+H(\xi,\tau),$$

$$\psi_2=\frac{c_0}{g}\frac{\partial H}{\partial\xi}-\frac{1}{g}(\frac{\partial G}{\partial\tau}+\frac{\partial F}{\partial\xi}\frac{\partial G}{\partial\xi})-\frac{c_1}{g}\frac{\partial F}{\partial\tau}-\frac{c_0h}{g}\psi_0\frac{\partial^3F}{\partial\xi^3}-\frac{h^2}{2g}(\frac{\partial^2F}{\partial\xi^2})^2, \tag{28}$$

where $H(\xi,\tau)$ is another unknown function whose governing equation will be obtained later. The use of the boundary condition $(23)_2$ yields the following evolution equation, which is the conventional Korteweg-deVries equation

$$\frac{\partial\psi_0}{\partial\tau}+\frac{3}{2}\frac{g}{c_0}\psi_0\frac{\partial\psi_0}{\partial\xi}+\frac{h^3g}{6c_0}\frac{\partial^3\phi_0}{\partial\xi^3}=0. \tag{29}$$

As is seen from this equation, the lowest order term in the perturbation expansion is governed by the Korteweg-de Vries equation.

Using the equation (29) in the equation (27), the function ψ_1 can be expressed as

$$\psi_1 = \frac{c_0}{g}\frac{\partial G}{\partial \xi} + \frac{\psi_0^2}{4h} + \frac{h^2}{6}\frac{\partial^2 \psi_0}{\partial \xi^2}. \tag{30}$$

In order to study the novelty of the present method we should study the higher order expansion. For that purpose we shall give the solution of $O(\epsilon^3)$ equations, which follows from (24)-(25)

$$\phi_3 = -\frac{1}{720}\frac{\partial^6 F}{\partial \xi^6}(z^6+6hz^5-40h^3z^3+96h^5z)+\frac{1}{24}\frac{\partial^4 G}{\partial \xi^4}(z^4+4hz^3-8h^3z)-\frac{1}{2}\frac{\partial^2 H}{\partial \xi^2}(z^2+2hz)+I(\xi,\tau), \tag{31}$$

where $I(\xi,\tau)$ is another unknown function whose governing equation will be obtained from the higher order expansions.

The use of the boundary condition $(25)_2$ yields the following evolution equation

$$\frac{2h^5}{15}\frac{\partial^6 F}{\partial \xi^6} + \frac{h^3}{3}\frac{\partial^4 G}{\partial \xi^4} + \psi_0\frac{\partial^2 G}{\partial \xi^2} + \psi_1\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial \psi_0}{\partial \xi}\frac{\partial G}{\partial \xi} + \frac{\partial F}{\partial \xi}\frac{\partial \psi_1}{\partial \xi} + c_1\frac{\partial \psi_0}{\partial \tau} + \frac{\partial \psi_1}{\partial \tau}+$$

$$\frac{c_0}{g}\frac{\partial^2 G}{\partial \xi \partial \tau} + \frac{c_0}{g}\frac{\partial^2 F}{\partial \xi^2}\frac{\partial G}{\partial \xi} + \frac{c_0}{g}\frac{\partial F}{\partial \xi}\frac{\partial^2 G}{\partial \xi^2} + \frac{c_0 c_1}{g}\frac{\partial^2 F}{\partial \xi \partial \tau} + h^2\psi_0\frac{\partial^4 F}{\partial \xi^4} + 2h^2\frac{\partial \psi_0}{\partial \xi}\frac{\partial^3 F}{\partial \xi^3} = 0. \tag{32}$$

Noting the relations

$$\frac{\partial F}{\partial \xi} = \frac{g}{c_0}\psi_0, \quad \frac{\partial G}{\partial \xi} = \frac{g}{c_0}\psi_1 - \frac{g}{4c_0 h}\psi_0^2 - \frac{gh^2}{6c_0}\frac{\partial^2 \psi_0}{\partial \xi^2}, \tag{33}$$

the following evolution equation is obtained

$$\frac{\partial \psi_1}{\partial \tau} + \frac{h^3 g}{6c_0}\frac{\partial^3 \psi_1}{\partial \xi^3} + \frac{3g}{2c_0}\frac{\partial}{\partial \xi}(\psi_0\psi_1) = \frac{\partial S(\psi_0)}{\partial \xi}, \tag{34}$$

where the non-homogeneous term $S(\psi_0)$ is defined by

$$S(\psi_0) = -\frac{19gh^5}{360c_0}\frac{\partial^4 \psi_0}{\partial \xi^4} - \frac{5gh^2}{12c_0}\psi_0\frac{\partial^2 \psi_0}{\partial \xi^2} - \frac{13gh^2}{48c_0}(\frac{\partial \psi_0}{\partial \xi})^2 + \frac{g}{8c_0 h}\psi_0^3 + \frac{3gc_1}{4c_0}\psi_0^2 + \frac{gh^3 c_1}{6c_0}\frac{\partial^2 \psi_0}{\partial \xi^2}. \tag{35}$$

Here, the equation (34) is the linearized Korteweg-deVries equation with non-homogeneous term $S(\psi_0)$ which contains the unknown coefficient c_1 which is to be determined from the solution.

2.2 Travelling Wave Solution

In this section we shall present a progressive wave solutions to equations (29) and (34). For that purpose we shall seek a solution to these equations in the following form

$$\psi_0 = F(\zeta), \quad \psi_1 = V(\zeta), \quad \zeta = \alpha(\xi - \beta\tau), \tag{36}$$

where α and β are two constants to be determined from the solutions. Introducing the proposed expression of $\psi_0(\zeta)$ into equation (29) we obtain

$$-\beta U' + \frac{3}{2}\frac{g}{c_0}UU' + \frac{h^3 g}{6c_0}\alpha^2 U''' = 0, \tag{37}$$

where a prime denotes the differentiation of the corresponding quantity with respect to ζ .

In this work we shall be concerned with the localized travelling wave solution, $i.e.$, U and its various order derivatives vanish as $\zeta \to \mp\infty$. Integrating equation (37) with respect to ζ and using the localization condition we obtain

$$-\beta U + \frac{3g}{4c_0}U^2 + \frac{h^3 g}{6c_0}\alpha^2 U'' = 0. \tag{38}$$

As is well-known this equation admits the progressive wave solution of the form

$$U = a\text{sech}^2\zeta, \tag{39}$$

where a is the amplitude of the solitary wave and the other quantities are defined by

$$\alpha = (\frac{3a}{4h^3})^{1/2}, \quad \beta = \frac{ga}{2c_0}. \tag{40}$$

Here, we note that the wave speed is proportional to the amplitude of the wave.

To obtain the solution for $V(\zeta)$, we introduce (36) and (39) into equations (34) and (35), which yields

$$-\beta V' + \frac{h^3 g}{6c_0}\alpha^2 V''' + \frac{3g}{2c_0}(UV)' = S(U)'. \tag{41}$$

Integrating this equation with respect to ζ and using the localization condition, we obtain

$$V'' + (12\text{sech}^2\zeta - 4)V = -\frac{6a^2}{h}\text{sech}^6\zeta + \frac{12a^2}{h}\text{sech}^4\zeta + (4ac_1 - \frac{19a^2}{5h})\text{sech}^2\zeta. \tag{42}$$

Here, we shall propose a solution for V of the following form

$$V = A\text{sech}^4\zeta + B\text{sech}^2\zeta, \tag{43}$$

where A and B are two constants to be determined from the solution of (42). Carrying out the derivative of V we have

$$V'' = -20A\text{sech}^6\zeta + (16A - 6B)\text{sech}^4\zeta + 4B\text{sech}^2\zeta \tag{44}$$

Inserting (43) and (44) into (42) and setting the coefficients of $\text{sech}^6\zeta$ and $\text{sech}^4\zeta$ equal to zero, one obtains

$$A = \frac{3a^2}{4h}, \quad B = \frac{a^2}{2h}, \tag{45}$$

and the remaining part of the equation (42) reads

$$(4c_1 a - \frac{19a^2}{5h})\text{sech}^2\zeta = 0. \tag{46}$$

If the constant c_1 was equal to zero, as in the classical reductive perturbation method, the equation would not be balanced. In order to balance the equation the solution must contain a secular term. In order to remove the secular term, the equation (46) must be satisfied identically; from which we obtain

$$c_1 = \frac{19a}{20h}. \tag{47}$$

Thus, in terms of real physical entities, the final solution takes the following form

$$\psi = a\,\text{sech}^2\zeta + \epsilon(\frac{3a^2}{4h}\,\text{sech}^4\zeta + \frac{a^2}{2h}\,\text{sech}^2\zeta). \tag{48}$$

with

$$\zeta = \epsilon^{1/2}\{(\frac{3a}{4h^3})^{1/2}[x - c_0(1 + \frac{a}{2h}\epsilon + \frac{19a^2}{40h^2}\epsilon^2)t]\} \tag{49}$$

3 Dressed solitary wave formalism for water waves

In order to see the novelty of the present formulation, we shall study the same problem by use of the method so called " the dressed solitary wave formalism ". For this purpose, we shall examine the solution to equations (5), (7) - (9) of the following form

$$\hat{\phi} = \hat{\phi}(\xi, z), \quad \hat{\psi} = \hat{\psi}(\xi), \quad \xi = k(x - Vt), \tag{50}$$

where k is the wave number and V is the speed of the wave. Introducing (50) into equations (5), (7) -(9) we have the field equation

$$k^2\frac{\partial^2\hat{\phi}}{\partial\xi^2} + \frac{\partial^2\hat{\phi}}{\partial z^2} = 0, \tag{51}$$

and the boundary conditions

$$\frac{\partial\hat{\phi}}{\partial z} = 0, \quad \text{at} \quad z = -h,$$

$$\frac{\partial\hat{\phi}}{\partial z} = -kV\frac{d\hat{\psi}}{d\xi} + k^2\frac{\partial\hat{\phi}}{\partial\xi}\frac{d\hat{\psi}}{d\xi}, \quad \text{on} \quad z = \hat{\psi}(\xi)$$

$$-kV\frac{\partial\hat{\phi}}{\partial\xi} + \frac{1}{2}[k^2(\frac{\partial\hat{\phi}}{\partial\xi})^2 + (\frac{\partial\hat{\phi}}{\partial z})^2] + g\hat{\psi} = 0, \quad \text{on} \quad z = \psi(\xi). \tag{52}$$

We shall further assume that the wave number is small (longwave) and the field variables $\hat{\phi}$ and $\hat{\psi}$, and the wave speed V can be expanded into a power series in terms of the wave number k as:

$$\hat{\phi} = k\phi_1 + k^3\phi_2 + k^5\phi_3 + k^7\phi_4 + ..., \quad \hat{\psi} = k^2\psi_1 + k^4\psi_2 + k^6\psi_3 + k^8\psi_4 + ...,$$

$$V = V_0(1 + k^2V_1 + k^4V_2 + ...) \tag{53}$$

Inserting the expansion (53) into the equations (51) and (52) and setting the coefficients of like powers of the wave number k equal to zero we obtain the following set of differential equations and the boundary conditions:

$O(k)$ equations

$$\frac{\partial^2\phi_1}{\partial z^2} = 0, \tag{54}$$

and the boundary conditions

$$\frac{\partial\phi_1}{\partial z} = 0, \quad \text{at} \quad z = 0, \quad \frac{\partial\phi_1}{\partial z} = \quad \text{at} \quad z = -h,$$

$$g\psi_1 - V_0\frac{\partial\phi_1}{\partial\xi}|_{z=0} = 0. \tag{55}$$

$O(k^3)$ equations

$$\frac{\partial^2\phi_2}{\partial z^2} + \frac{\partial^2\phi_1}{\partial\xi^2} = 0, \tag{56}$$

and the boundary conditions

$$\frac{\partial\phi_2}{\partial z} = 0, \quad \text{at} \quad z = -h, \quad \frac{\partial\phi_2}{\partial z}|_{z=0} + V_0\frac{d\psi_1}{d\xi} = 0,$$

$$g\psi_2 - V_0\frac{\partial\phi_2}{\partial\xi}|_{z=0} - V_0V_1\frac{\partial\phi_1}{\partial\xi}|_{z=0} + \frac{1}{2}(\frac{\partial\phi_1}{\partial\xi})^2|_{z=0} = 0. \tag{57}$$

$O(k^5)$ equations

$$\frac{\partial^2\phi_3}{\partial z^2} + \frac{\partial^2\phi_2}{\partial\xi^2} = 0, \tag{58}$$

$$(\frac{\partial\phi_3}{\partial z} + \psi_1\frac{\partial^2\phi_2}{\partial z^2})|_{z=0} + V_0(\frac{d\psi_2}{d\xi} + V_1\frac{d\psi_1}{d\xi}) - \frac{\partial\phi_1}{\partial\xi}|_{z=0}\frac{d\psi_1}{d\xi} = 0,$$

$$g\psi_3 - \big[V_0(\frac{\partial\phi_3}{\partial\xi} + \psi_1\frac{\partial^2\phi_2}{\partial\xi\partial z}) + V_0V_1\frac{\partial\phi_2}{\partial\xi} + V_0V_2\frac{\partial\phi_1}{\partial\xi}\big]|_{z=0} + (\frac{\partial\phi_1}{\partial\xi}\frac{\partial\phi_2}{\partial\xi})|_{z=0} + \frac{1}{2}(\frac{\partial\phi_2}{\partial z})^2|_{z=0} = 0. \tag{59}$$

$O(k^7)$ equations

$$\frac{\partial^2\phi_4}{\partial z^2} + \frac{\partial^2\phi_3}{\partial\xi^2} = 0, \tag{60}$$

and the boundary conditions

$$\frac{\partial\phi_4}{\partial z} = 0, \quad \text{at} \quad z = -h,$$

$$[\frac{\partial\phi_4}{\partial z}+\psi_1\frac{\partial^2\phi_3}{\partial z^2}+\frac{1}{2}\psi_1^2\frac{\partial^3\phi_2}{\partial z^3}+\psi_2\frac{\partial^2\phi_2}{\partial z^2}]|_{z=0}+V_0(\frac{d\psi_3}{d\xi}+V_1\frac{d\psi_2}{d\xi}+V_2\frac{d\psi_1}{d\xi})-[\frac{\partial\phi_1}{\partial\xi}\frac{d\psi_2}{d\xi}+\frac{\partial\phi_2}{\partial\xi}\frac{d\psi_1}{d\xi}]|_{z=0}=0,$$

$$g\psi_4-[V_0(\frac{\partial\phi_4}{\partial\xi}+V_1\frac{\partial^2\phi_3}{\partial\xi\partial z}+\frac{1}{2}\psi_1^2\frac{\partial^3\phi_2}{\partial\xi\partial z^2}+\psi_2\frac{\partial^2\phi_2}{\partial\xi\partial z})-V_0V_1(\frac{\partial\phi_3}{\partial\xi}+\psi_1\frac{\partial^2\phi_2}{\partial\xi\partial z})-V_0V_2\frac{\partial\phi_2}{\partial\xi}-V_0V_3\frac{\partial\phi_1}{\partial\xi}]|_{z=0}$$

$$+[\frac{1}{2}(\frac{\partial\phi_2}{\partial\xi})^2+\frac{\partial\phi_1}{\partial\xi}(\frac{\partial\phi_3}{\partial\xi}+\psi_1\frac{\partial^2\phi_2}{\partial\xi\partial z})]|_{z=0}+\frac{\partial\phi_2}{\partial z}(\frac{\partial\phi_3}{\partial z}+\psi_1\frac{\partial^2\phi_2}{\partial z^2})|_{z=0}=0. \quad (61)$$

3.1 Solution of the field equations

From the solution of the set given in equations (54) and (55) we obtain

$$\phi_1=F(\xi),\quad \psi_1=\frac{V_0}{g}\frac{dF}{d\xi}, \quad (62)$$

where $F(\xi)$ is an arbitrary function of its argument and to be determined later.

Similarly, from the solution of the equations (56) and (57) one has

$$\phi_2=-\frac{1}{2}\frac{d^2F}{d\xi^2}(z^2+2hz)+G(\xi),\quad V_0=(gh)^{1/2},\quad \psi_2=\frac{V_0}{g}\frac{dG}{d\xi}+\frac{V_0V_1}{g}\frac{dF}{d\xi}-\frac{1}{2g}(\frac{dF}{d\xi})^2, \quad (63)$$

where $G(\xi)$ is another unknown function whose governing equation will be obtained later.

From the solution of equations (58) and (59) ($O(k^5)$ order equations) we have

$$\phi_3=\frac{1}{24}\frac{d^4F}{d\xi^4}(z^4+4hz^3-8h^3z)-\frac{1}{2}\frac{d^2G}{d\xi^2}(z^2+2hz)+H(\xi),$$

$$\psi_3=\frac{V_0}{g}\frac{dH}{d\xi}-h\psi_1\frac{d\psi_1}{d\xi}+V_1\psi_2+\frac{3V_1}{2h}\psi_1^2+(V_2-V_1^2)\psi_1-\frac{1}{h}\psi_1\psi_2-\frac{1}{2h^2}\psi_1^3-\frac{h}{2}(\frac{d\psi_1}{d\xi})^2. \quad (64)$$

The use of the boundary condition $(58)_2$ yields

$$\frac{d^4F}{d\xi^4}+\frac{9V_0}{gh^3}\frac{dF}{d\xi}\frac{d^2F}{d\xi^2}-\frac{6V_1}{h^2}\frac{d^2F}{d\xi^2}=0, \quad (65)$$

or, using the equation (62) we obtain the governing equation for ψ_1 as

$$\frac{d^3\psi_1}{d\xi^3}+\frac{9}{h^3}\psi_1\frac{d\psi_1}{d\xi}-\frac{6V_1}{h^2}\frac{d\psi_1}{d\xi}=0. \quad (66)$$

In the present work, we shall be concerned with the localized travelling waves, *i.e.,* the function ψ_1 and its various order derivatives vanish as $\xi\to\pm\infty$. Hence, integrating (66) with respect to ξ and using the localization condition we obtain

$$\frac{d^2\psi_1}{d\xi^2}+\frac{9}{2h^3}\psi_1^2-\frac{6V_1}{h^2}\psi_1=0. \quad (67)$$

We shall seek a solution to equation (67) of the following form

$$\psi_1=a\,\mathrm{sech}^2\eta,\quad \eta=\alpha\xi \quad (68)$$

where a is the wave amplitude and α is a constant to be determined later. Introducing (68) into equation (67) and setting the coefficients of like powers of $\mathrm{sech}\eta$ equal to zero, we obtain

$$\alpha=(\frac{3a}{4h^3})^{1/2},\quad V_1=\frac{a}{2h}. \quad (69)$$

From the solution of equations (60) and (61) one has

$$\phi_4=-\frac{1}{720}\frac{d^6F}{d\xi^6}(z^6+6hz^5-40h^3z^3+96h^5z)+\frac{1}{24}\frac{d^4G}{d\xi^4}(z^4+4hz^3-8h^3z)-\frac{1}{2}\frac{d^2H}{d\xi^2}(z^2+2hz)+I(\xi) \quad (70)$$

where $I(\xi)$ is another unknown function to be determined from the solution of the higher order expansions. The use of the boundary condition $(61)_2$ yields

$$\frac{d^3\psi_2}{d\xi^3} - \frac{6V_1}{h^2}\frac{d\psi_2}{d\xi} + \frac{9}{h^3}\frac{d}{d\xi}(\psi_1\psi_2) = \frac{dS(\psi_1)}{d\xi}, \tag{71}$$

where the function $S(\psi_1)$ is defined by

$$S(\psi_1) = -\frac{2h^2}{5}\frac{d^4\psi_1}{d\xi^4} - \frac{5}{2h}(\psi_1')^2 - \frac{4}{h}\psi_1\frac{d^2\psi_1}{d\xi^2} + V_1\frac{d^2\psi_1}{d\xi^2} - \frac{3}{h^4}\psi_1^3 + \frac{15V_1}{2h^3}\psi_1^2 + \frac{3}{h^2}(2V_2 - V_1^2)\psi_1. \tag{72}$$

Integrating the equation (71) with respect to ξ and utilizing the localization condition we have,

$$\frac{d^2\psi_2}{d\xi^2} - \frac{6V_1}{h^2}\psi_2 + \frac{9}{h^3}(\psi_1\psi_2) = S(\psi_1). \tag{73}$$

In order to use the solution proposed in equation (68) it would be convenient to express the differential equation (73) in terms of η. The result will be as follows:

$$\psi_2'' + (\frac{12\psi_1}{a} - 4)\psi_2 = \bar{S}(\psi_1), \tag{74}$$

where a prime denotes the differentiation with respect to η and $\bar{S}(\psi_1)$ is defined by

$$\bar{S}(\psi_1) = -\frac{3a}{10h}\psi_1^{(4)} - \frac{5}{2h}(\psi_1')^2 - \frac{4}{h}\psi_1\psi_2'' + \frac{a}{2h}\psi_1'' - \frac{4}{ah}\psi_1^3 + \frac{5}{h}\psi_1^2 + \frac{4h}{a}(2V_2 - \frac{a^2}{4h^2})\psi_1. \tag{75}$$

Introducing the solution (68) into (74) and (75) the following differential equation is obtained

$$\psi_2'' + (12\text{sech}^2\eta - 4)\psi_2 = -\frac{6a^2}{h}\text{sech}^6\eta + \frac{12a^2}{h}\text{sech}^4\eta(8hV_2 - \frac{19a^2}{5h})\text{sech}^2\eta \tag{76}$$

To obtain the travelling wave solution to this order of equation, we shall propose the function ψ_2 as

$$\psi_2 = A\,\text{sech}^4\eta + B\,\text{sech}^2\eta \tag{77}$$

where A and B are two constants to be determined from the solution of (76). Carrying out the derivative of ψ_2 we have

$$\psi_2'' = -20A\text{sech}^6\eta + (16A - 6B)\text{sech}^4\eta + 4B\text{sech}^2\eta. \tag{78}$$

Inserting (78) into equation (76) and setting the coefficients of $\text{sech}^6\eta$ and $\text{sech}^4\eta$ equal to zero, one obtains

$$A = \frac{3a^2}{4h}, \quad B = \frac{a^2}{2h}. \tag{79}$$

The remaining part of the equation (76) reads

$$(8hV_2 - \frac{19a^2}{5h}\text{sech}^2\eta = 0 \tag{80}$$

If the constant V_2 was equal to zero, the equation would not be balanced and the solution will contain a secular term. In order to remove the secular term, the equation (80) must be satisfied identically. Thus, we obtain

$$V_2 = \frac{19a^2}{40h^2}. \tag{81}$$

Hence, in terms of the real physical entities, the final solution up to this order may be given as follows:

$$\hat{\psi} = k^2a\,\text{sech}^2\eta + k^4(\frac{3a^2}{4h}\text{sech}^4\eta + \frac{a^2}{2h^2}\text{sech}^2\eta), \tag{82}$$

with

$$\eta = k(\frac{3a}{4h^3})^{1/2}[x - V_0(1 + \frac{a}{2h}k^2 + \frac{19a^2}{40h^2}k^4)t]. \tag{83}$$

This solution is exactly the same with the one given in equations (48) and (49).

As might be seen from these formulations, the correction term to the soliton speed can be obtained quite easily. For instance, the $O(\epsilon^2)$ order correction term is $\frac{19a^2}{40h^2}$, which corresponds to $1/A_2$ in Kraenkel et al[6] formulation, wherein A_2 remains to be undetermined. By use of this formulation the higher order corrections term can be obtained without any principal difficulty.

4 Concluding Remarks

The study of the effects of higher order terms in the perturbation expansion of field quantities through the use of classical reductive perturbation method leads to some secularities. To eliminate such secularities various methods, like renormalization method of Kodama and Taniuti [4], the multiple scale expansion method by Kraenkel et al[6], have been presented in the current literature. The results of present work and of those given in references [8] and [9] proved that the "modified reductive perturbation method" presented by us is the most simplest and the effective one. By use of this method, any order of correction term may be obtained without any principal difficulties.

References

[1] N. Davidson: Methods in nonlinear plasma theory. *Academic Press*. New York(1972)

[2] T. Taniuti: Reductive perturbation method and far field of wave equations. *Progress in Theoretical Physics Supplement* .**55**: 1-35(1974)

[3] N. Sugimoto , T. Kakutani: Note on higher order terms in reductive perturbation theory. *J. Phys. Soc. Japan*. 43: 1469-1470(1977)

[4] Y. Kodama , T. Taniuti: Higher order approximation in reductive perturbation method 1, Weakly dispersive system. *J. Phys. Soc. Japan*. 45: 298- 310(1978)

[5] H. Washimi , T. Taniuti: Propagation of ionic-acoustic solitary waves of small amplitude. *Phys. Rev. Letter*. 17: 996- 998(1966)

[6] R. A. Kraenkel: M. A. Manna , J. G. Pereira: The Korteweg-deVries hierarchy and long water waves. *J. Math. Phys.*. 36: 307- 320(1995)

[7] M. Malfliet , E. Wieers: The theory of nonlinear ion-acoustic waves revisited. *J. Plasma Phys.*. 56: 441-453(1996)

[8] H. Demiray: A modified reductive perturbation method as applied to nonlinear ion-acoustic waves. *J. Phys. Soc. Japan.* 68: 1833- 1837(1999)

[9] H. Demiray: On the contribution of higher order terms to solitary waves in fluid-filled elastic tubes. *Z. Agnew. Math. Phys. (ZAMP)*. 51: 75-91(2000)

[10] L. Wang, J. Zhou , L. Ren: The exact solitary wave solutions for a family of BBM equation. *Int. J. Nonlinear Sci.* 1: 58-64(2006)

[11] D. Lu, B. Hong , L. Tian: Backlund transformation and N-soliton solution to the combined KdV-Burgers equation with variable coefficien. *Int. J. Nonlinear Sci.*2:3-10(2006)

[12] A. Bekir , E. Yusufoglu: On the extended Tanh method applications of nonlinear equations, *Int. J. Nonlinear Sci.* 4: 10-16(2007)

[13] L. Wang: New exact solution for MNV equation. *Int. J. Nonlinear Sci.* 5: 152-157(2008)