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On the Bifurcation of Traveling Wave Solution of Generalized Camassa-Holm Equation

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Abstract: The generalized Camassa-Holm equation

$$u_t + 2ku_x - u_{xxt} + au^2u_x = 2u_xu_{xx} + uu_{xxx} + \gamma u_{xxx}$$

is considered in this paper. Under traveling wave variable substitution, the equation is related to a planar singular system. By making a transformation this singular system becomes a regular system. Through discussing the dynamical behavior of the regular system, the explicit periodic blow-up solutions and solitary wave solutions of the generalized Camassa-Holm equation are obtained. Also the limit forms of those solutions are presented.

Key words: generalized Camassa-Holm equation; dynamical behavior; explicit solutions; periodic blow-up; solitary wave

1 Introduction

Camassa and Holm [1] derived a shallow water wave equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \tag{1.1}$$

where u is the fluid velocity in the x direction (or equivaqlently the height of the water's free surface above a flat bottom), k is a constant related to the critical shallow water wave speed. For $k = 0$, they showed that Eq.(1.1) has peakons of the form $u(x,t) = ce^{-|x-ct|}$. For the case of $k \neq 0$ and the wave speed $c = \frac{k}{2}$, Liu and Qian [2] gave three ways to seek the peakon of Eq.(1.1). For any parameter k and constant wave speed c , Liu etc.[3] showed that Eq.(1.1) has peakons of the form

$$u(x,t) = (k+c)e^{-|x-ct|} - k, \tag{1.2}$$

which can be seen as a weak solution being similar to that in [4-6]. In Ref.[7] Tian et al discussed the traveling wave solutions and double soliton solutions of Eq.(1.1), and introduced the definitions of concave, convex peaked soliton and smooth soliton solution.

In 2001, Dullin, Gottwald and Holm[8] presented a nonlinear equation

$$u_t + c_0u_x + 3uu_x - \alpha^2(u_{xxt} + uu_{xxx} + 2u_xu_{xx}) + \gamma u_{xxx} = 0, \tag{1.3}$$

which is called CH-γ equation. Clearly, when $\alpha^2 = 1$ and $\gamma = 0$, Eq.(1.3) becomes Eq.(1.1). In [9-11], it was shown that Eqs.(1.1) and (1.3) have many similar properties.

In 2001, Liu and Qian[12] suggested a generalized Comassa-Holm equation

$$u_t + 2ku_x - u_{xxt} + au^mu_x = 2u_xu_{xx} + uu_{xxx}. \tag{1.4}$$

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Clearly, when $m = 1$, Eq.(1.4) becomes Eq.(1.1). In [13] Tian and Song gave some new peaked solitary wave solutions for Eq.(1.4) when $m = 1, 2, 3$. Khuri[14] gave some explicit expressions of the peakons and discontinuous solitary waves for Eq.(1.4) when $m = 1, 2, 3$. Tian and Yin[15,16] found that Eq.(1.4) has other forms of solitary patterns solutions in different conditions. Wang et al[17] obtained compacton solutions of a family of BBM equation. Fan et al[18] obtained new compactons in nonlinear atomic chain equations with first and second neighbor interactions. Ju [19] obtained peakon solutions to the Dullin-Gottwald-Holm Equation. And Tian et al[20] studied boundary control of viscous Dullin-Gottwald-Holm equation .

When $m = 2$, Eq.(1.4) becomes

$$u_t + 2ku_x - u_{xxt} + au^2u_x = 2u_xu_{xx} + uu_{xxx}. \tag{1.5}$$

For case of $a = 3$ and $k = 0$, using several special functions, Wazwaz[21,22] obtained many explicit solitary wave solutions.

Liu and Ouyang[23] showed that the bell-shaped solitary wave and peakon coexist in Eq.(1.5) when $a = 3$ and $k = 0$.

In this paper we consider the following equation

$$u_t + 2ku_x - u_{xxt} + au^2u_x = 2u_xu_{xx} + uu_{xxx} + \gamma u_{xxx}. \tag{1.6}$$

Through some special phase orbits, a class of explicit periodic wave solutions are obtained. Since such solutions blow up periodically, they are called periodic blow-up solutions. Also the limit forms of these solutions are got. And by using the theory of planar dynamical systems to a variant of Eq.(1.6), the existence of solitary wave solutions, the periodic cusp wave solutions, the kink wave solutions and the one-sided breaking wave solutions are proved.

In order to state our main results conveniently, for given constant $c > 0$, let

$$s_1 : k = s_1(a, \gamma) = -\frac{(c-\gamma)^2}{6}a + \frac{c}{2}, \tag{1.7}$$

$$s_2 : k = s_2(a, \gamma) = -\frac{(c-\gamma)^2}{12}a + \frac{c}{2}, \tag{1.8}$$

$$s_3 : k = s_3(a, \gamma) = \frac{c}{2}, \tag{1.9}$$

$$\alpha = \sqrt{\tfrac{6c-12k-a(c-\gamma)^2}{a}} \quad \text{for} \quad \frac{6c-12k-a(c-\gamma)^2}{a} \geq 0, \tag{1.10}$$

$$\beta_1 = \sqrt{\tfrac{|a|\alpha}{12}}, \tag{1.11}$$

$$\beta_2 = \sqrt{\tfrac{|a(c-\gamma)|}{12}}, \tag{1.12}$$

$$\beta_3 = \sqrt{\tfrac{|a|(\alpha+|c-\gamma|)}{24}}, \tag{1.13}$$

$$\beta_4 = \sqrt{\tfrac{|a(c-\gamma)|}{24}}. \tag{1.14}$$

$snz = sn(z, l)$ be the Jacobian elliptic function with modulus l, $\sec z$ and $\csc z$ trigonometric functions, $\coth z$ and $\cosh z$ be hyperbolic functions. On the parametric space $a - \gamma - k$, we mark the $s_i (i = 1, 2, 3)$ and regions (A_j),$(B_j)(j = 1, 2, ..., 8)$ surrounded by $s_i(i = 1, 2, 3)$ and γOk- coordinate and plane: $\gamma = c$ as Fig.1.

$$\begin{cases} (A_1) : a > 0, \gamma > c, k < -\frac{(c-\gamma)^2}{6}a + \frac{c}{2} \\ (A_2) : a > 0, \gamma > c, -\frac{(c-\gamma)^2}{6}a + \frac{c}{2} < k < -\frac{(c-\gamma)^2}{12}a + \frac{c}{2} \\ (A_3) : a > 0, \gamma > c, -\frac{(c-\gamma)^2}{12}a + \frac{c}{2} < k < \frac{c}{2} \\ (A_4) : a > 0, \gamma > c, k > \frac{c}{2} \\ (A_5) : a < 0, \gamma > c, k > -\frac{(c-\gamma)^2}{6}a + \frac{c}{2} \\ (A_6) : a < 0, \gamma > c, -\frac{(c-\gamma)^2}{12}a + \frac{c}{2} < k < -\frac{(c-\gamma)^2}{6}a + \frac{c}{2} \\ (A_7) : a < 0, \gamma > c, \frac{c}{2} < k < -\frac{(c-\gamma)^2}{12}a + \frac{c}{2} \\ (A_8) : a < 0, \gamma > c, k < \frac{c}{2} \end{cases}$$

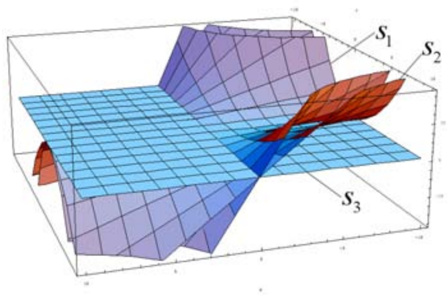


Figure 1: The graphs of $s_i(i=1,2,3)$ and regions $(A_j),(B_j)(j=1,2,...,8)$ on $a-\gamma-k$ space.

$$\begin{cases} (B_1): a<0, \gamma<c, k>-\frac{(c-\gamma)^2}{6}a+\frac{c}{2} \\ (B_2): a<0, \gamma<c, -\frac{(c-\gamma)^2}{12}a+\frac{c}{2}<k<-\frac{(c-\gamma)^2}{6}a+\frac{c}{2} \\ (B_3): a<0, \gamma<c, \frac{c}{2}<k<-\frac{(c-\gamma)^2}{12}a+\frac{c}{2} \\ (B_4): a<0, \gamma<c, k<\frac{c}{2} \\ (B_5): a>0, \gamma<c, k<-\frac{(c-\gamma)^2}{6}a+\frac{c}{2} \\ (B_6): a>0, \gamma<c, -\frac{(c-\gamma)^2}{6}a+\frac{c}{2}<k<-\frac{(c-\gamma)^2}{12}a+\frac{c}{2} \\ (B_7): a>0, \gamma<c, -\frac{(c-\gamma)^2}{12}a+\frac{c}{2}<k<\frac{c}{2} \\ (B_8): a>0, \gamma<c, k>\frac{c}{2} \end{cases}$$

Using the notations above, we discuss bifurcation phase portraits of system (1.6) and state our main results in Section 2. In Section 3, we prove our main results: Theorems 1-4 and give a concrete example. A short conclusion will be given in Section 4.

2 Planar dynamical system of Eq.(1.6) and its bifurcation phase portraits

For given constant c, substituting

$$\xi = x - ct, \tag{2.1}$$

and $u=\phi(\xi)$ into Eq.(1.6), it follows that

$$-c\phi' + 2k\phi' + c\phi''' + a\phi^2\phi' = 2\phi'\phi'' + \phi\phi''' + \gamma\phi''', \tag{2.2}$$

where "$'$" is the derivative with respect to ξ .

Integrating (2.2) once with respect to ξ and let integral constant be zero, we have

$$\phi''(\phi - c + \gamma) = (2k-c)\phi + \frac{a}{3}\phi^3 - \frac{(\phi')^2}{2}. \tag{2.3}$$

Using (2.3), we establish the following planar system

$$\begin{cases} \frac{d\phi}{d\xi} = y \\ \frac{dy}{d\xi} = \frac{\frac{a}{3}\phi^3 + (2k-c)\phi - \frac{1}{2}y^2}{\phi - c + \gamma} \end{cases}. \tag{2.4}$$

Then ,system (2.4) has three elementary critical points at $(0,0)$, $(\pm\sqrt{\frac{3(c-2k)}{a}},0)$ if $\frac{3(c-2k)}{a}>0$. In the straight line $\phi=c-\gamma$ of the (ϕ,y)-phase plane, the second equation of (2.4) is discontinuous. We call $\phi=c-\gamma$ as a singular straight line in which there exist two critical points of (2.4) at $(c-\gamma,\pm\sqrt{Y})$ if $Y=2(c-\gamma)[\frac{a}{3}(c-\gamma)^2+(2k-c)]>0$. For avoiding the inconvenice temporarily we make transformation

$$d\tau = \frac{d\xi}{\phi - c + \gamma}. \tag{2.5}$$

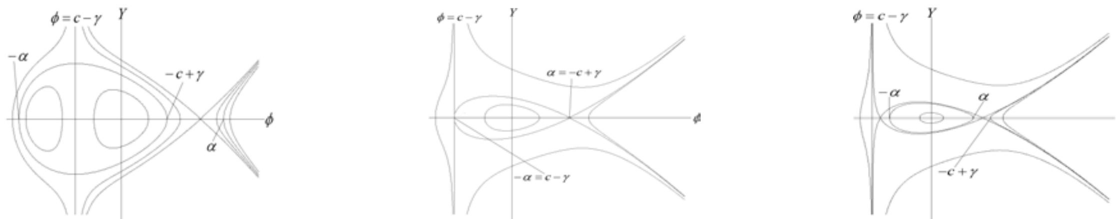
Under the transformation (2.5), system (2.4) becomes

$$\begin{cases} \frac{d\phi}{d\tau} = y(\phi - c + \gamma) \\ \frac{dy}{d\tau} = \frac{a}{3}\phi^3 + (2k - c)\phi - \frac{1}{2}y^2 \end{cases} \tag{2.6}$$

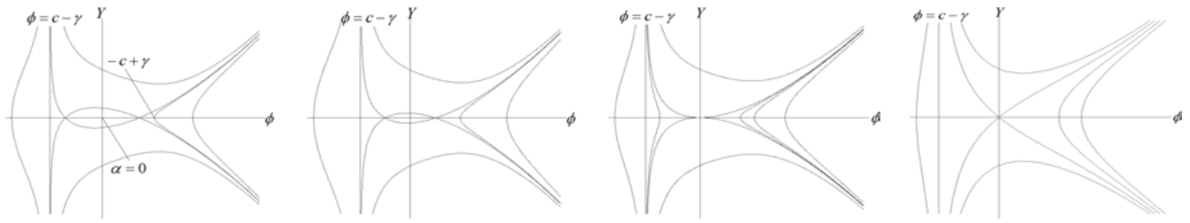
Since both (2.4) and (2.6) have the same first integral

$$y^2(\phi - c + \gamma) - \frac{a}{6}\phi^4 - (2k - c)\phi^2 = h, \tag{2.7}$$

the two system have the same topological phase portraits except the line $\phi = c - \gamma$. Though qualitative analysis, we draw the bifurcation phase portraits as Figs.2-5.

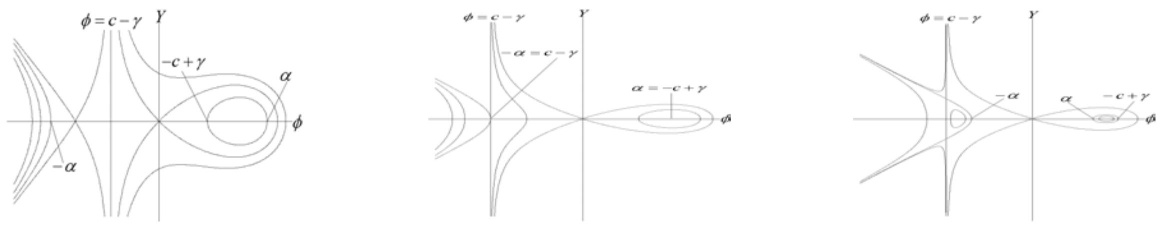


(a) $k < -\frac{(c-\gamma)^2}{6}a + \frac{c}{2}$ (b) $k = -\frac{(c-\gamma)^2}{6}a + \frac{c}{2}$ (c) $-\frac{(c-\gamma)^2}{6}a + \frac{c}{2} < k < -\frac{(c-\gamma])^2}{12}a + \frac{c}{2}$

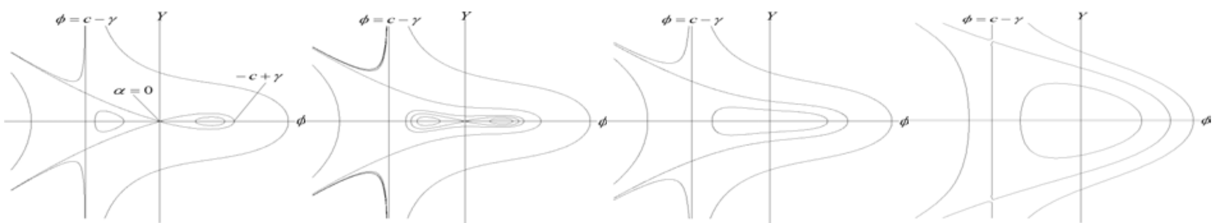


(d) $k = -\frac{(c-\gamma)^2}{12}a + \frac{c}{2}$ (e) $-\frac{(c-\gamma)^2}{12}a + \frac{c}{2} < k < \frac{c}{2}$ (f) $k = \frac{c}{2}$ (g) $k > \frac{c}{2}$

Figure 2: Bifurcation phase portraits of system (2.4) and (2.6) when $a > 0$ and $\gamma > c$



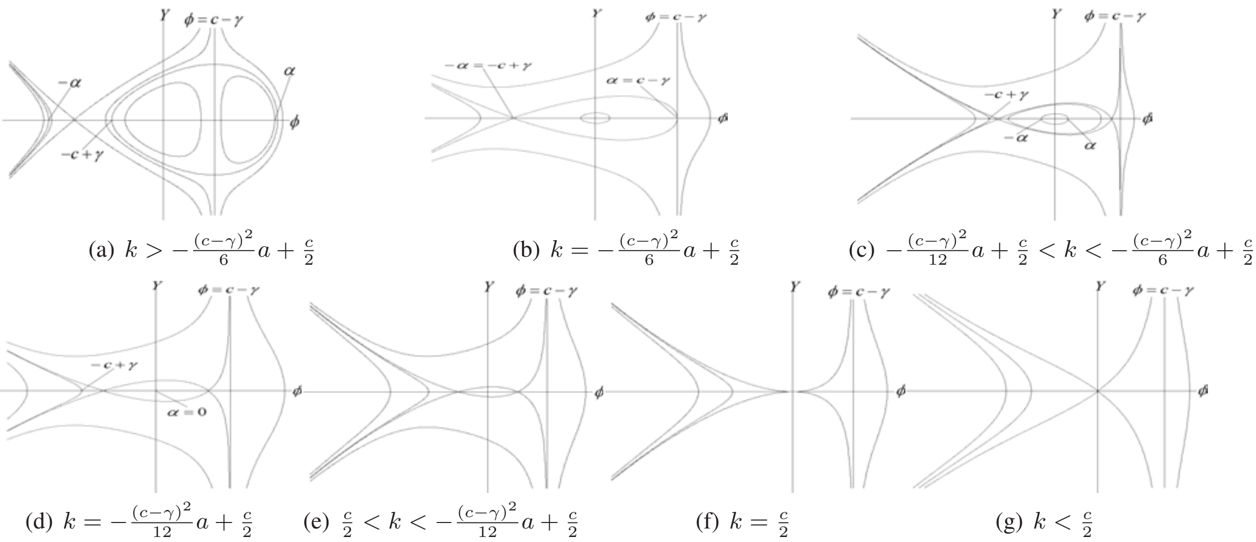
(a) $k > -\frac{(c-\gamma)^2}{6}a + \frac{c}{2}$ (b) $k = -\frac{(c-\gamma)^2}{6}a + \frac{c}{2}$ (c) $-\frac{(c-\gamma)^2}{12}a + \frac{c}{2} < k < -\frac{(c-\gamma)^2}{6}a + \frac{c}{2}$



(d) $k = -\frac{(c-\gamma)^2}{12}a + \frac{c}{2}$ (e) $\frac{c}{2} < k < -\frac{(c-\gamma)^2}{12}a + \frac{c}{2}$ (f) $k = \frac{c}{2}$ (g) $k < \frac{c}{2}$

Figure 3: Bifurcation phase portraits of system (2.4) and (2.6) when $a < 0$ and $\gamma > c$.

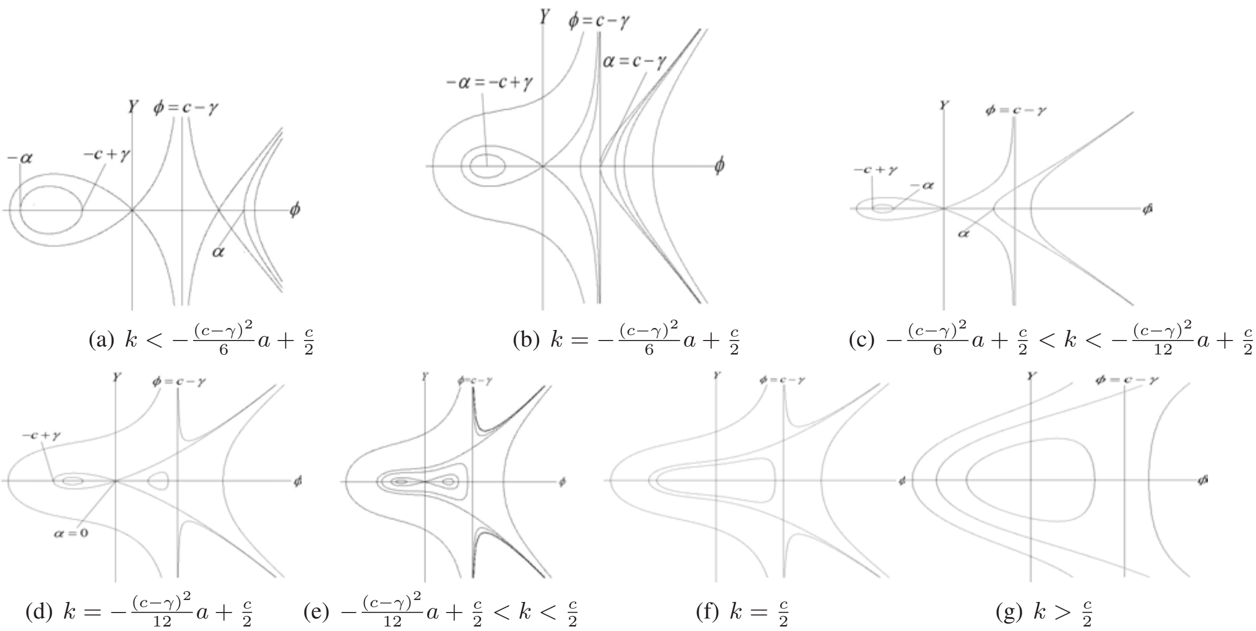
Remark 1 *: When $(a, \gamma, k) \in (A_1)$, (A_2), (A_5), (A_6), (B_1), (B_2), (B_5), (B_6), s_1 and s_2, there exist periodic blow-up and blow-up solutions. And the expression of those blow-up solutions are stated in the* *following Theorems 1 and2. When $(a, \gamma, k) \in (A_3)$ and (B_3), there exist kink wave solutions. When $(a, \gamma, k) \in (A_4)$ and (B_4), there exist two-sided breaking wave solutions. When $(a, \gamma, k) \in (A_7)$ and (B_7), there exist solitary wave solutions of valley type and solitary wave solutions of peak type. When $(a, \gamma, k) \in (A_8)$ and (B_8), there exist periodic cusp wave solutions of peak type.*



(a) $k > -\frac{(c-\gamma)^2}{6}a + \frac{c}{2}$ (b) $k = -\frac{(c-\gamma)^2}{6}a + \frac{c}{2}$ (c) $-\frac{(c-\gamma)^2}{12}a + \frac{c}{2} < k < -\frac{(c-\gamma)^2}{6}a + \frac{c}{2}$

(d) $k = -\frac{(c-\gamma)^2}{12}a + \frac{c}{2}$ (e) $\frac{c}{2} < k < -\frac{(c-\gamma)^2}{12}a + \frac{c}{2}$ (f) $k = \frac{c}{2}$ (g) $k < \frac{c}{2}$

Figure 4: Bifurcation phase portraits of system (2.4) and (2.6) when $a < 0$ and $\gamma < c$.



(a) $k < -\frac{(c-\gamma)^2}{6}a + \frac{c}{2}$ (b) $k = -\frac{(c-\gamma)^2}{6}a + \frac{c}{2}$ (c) $-\frac{(c-\gamma)^2}{6}a + \frac{c}{2} < k < -\frac{(c-\gamma)^2}{12}a + \frac{c}{2}$

(d) $k = -\frac{(c-\gamma)^2}{12}a + \frac{c}{2}$ (e) $-\frac{(c-\gamma)^2}{12}a + \frac{c}{2} < k < \frac{c}{2}$ (f) $k = \frac{c}{2}$ (g) $k > \frac{c}{2}$

Figure 5: Bifurcation phase portraits of system (2.4) and (2.6) when $a > 0$ and $\gamma < c$.

Theorem 1 *For given constant $c > 0$, $\gamma > c$ and parametric regions marked in Fig.1, about the solutions of Eq.(1.6) we have:*

(1) If $(a, \gamma, k) \in (A_1)$, then there is a periodic blow-up solution

$$u_1(x,t) = \alpha(2sn^{-2}\beta_1(x - ct) - 1), \tag{2.8}$$

where the modulus of sn *is*

$$k_1 = \sqrt{\tfrac{\alpha+|c-\gamma|}{2\alpha}}. \tag{2.9}$$

(2) If $a > 0$, $\gamma > c$ *and* $(a, \gamma, k) \in s_1$, *then there is a blow-up solution*

$$u_2(x,t) = (c-\gamma)(2\coth^2\beta_2(x-ct)-1). \tag{2.10}$$

(3) If $(a, \gamma, k) \in (A_2)$ *, then there is a periodic blow-up solution*

$$u_3(x,t) = (\alpha+|c-\gamma|)sn^{-2}\beta_3(x-ct)-\alpha, \tag{2.11}$$

where the modulus of sn *is*

$$k_2 = \sqrt{\tfrac{2\alpha}{\alpha+|c-\gamma|}}. \tag{2.12}$$

(4) If $a > 0$, $\gamma > c$ *and* $(a, \gamma, k) \in s_2$ *then there is a periodic blow-up solution*

$$u_4(x,t) = (c-\gamma)\csc^2\beta_4(x-ct). \tag{2.13}$$

(5) If $(a, \gamma, k) \in (A_5)$ *, then there is a periodic blow-up solution*

$$u_5(x,t) = \alpha(1-2sn^{-2}\beta_1(x-ct)), \tag{2.14}$$

where the modulus of sn *is*

$$k_3 = \sqrt{\tfrac{\alpha-|c-\gamma|}{2\alpha}}. \tag{2.15}$$

(6) If $a < 0$, $\gamma > c$ *and* $(a, \gamma, k) \in s_1$ *, then there is a periodic blow-up solution*

$$u_6(x,t) = (c-\gamma)(1-2\csc^2\beta_2(x-ct)). \tag{2.16}$$

(7) If $(a, \gamma, k) \in (A_6)$ *, then there is a periodic blow-up solution*

$$u_7(x,t) = |c-\gamma|-(\alpha+|c-\gamma|)sn^{-2}\beta_3(x-ct), \tag{2.17}$$

where the modulus of sn *is*

$$k_4 = \sqrt{\tfrac{|c-\gamma|-\alpha}{|c-\gamma|+\alpha}}. \tag{2.18}$$

(8) If $a < 0$, $\gamma > c$ *and* $(a, \gamma, k) \in s_2$, *then there is a periodic blow-up solution*

$$u_8(x,t) = -(c-\gamma)\csc h^2\beta_4(x-ct). \tag{2.19}$$

Theorem 2 *For given constant* $c > 0$, $\gamma < c$ *and parametric regions marked in Fig.1, about the solutions of Eq.(1.6) we have:*

(1) If $(a, \gamma, k) \in (B_1)$ *, then there is a periodic blow-up solution* $-u_1(x, t)$.
(2) If $a < 0$, $\gamma < c$ *and* $(a, \gamma, k) \in s_1$, *then there is a blow-up solution* $u_2(x, t)$.
(3) If $(a, \gamma, k) \in (B_2)$ *, then there is a periodic blow-up solution* $-u_3(x, t)$.
(4) If $a < 0$, $\gamma < c$ *and* $(a, \gamma, k) \in s_2$ *then there is a periodic blow-up solution* $u_4(x, t)$.
(5) If $(a, \gamma, k) \in (B_5)$ *, then there is a periodic blow-up solution* $-u_5(x, t)$.
(6) If $a > 0$, $\gamma < c$ *and* $(a, \gamma, k) \in s_1$ *, then there is a periodic blow-up solution* $u_6(x, t)$.
(7) If $(a, \gamma, k) \in (B_6)$ *, then there is a periodic blow-up solution* $-u_7(x, t)$.
(8) If $a > 0$, $\gamma < c$ *and* $(a, \gamma, k) \in s_2$, *then there is a periodic blow-up solution* $u_8(x, t)$.

Theorem 3 *For given constant* $c > 0$ *and* $\gamma > c$, *the solution* $u_i(\xi)(i = 1, \ldots, 8)$ *have the following relations:*

(1) If $(a, \gamma, k) \in (A_1)$ *and tends to* s_1 *, then* $-u_1(x, t)$ *becomes* $u_2(x, t)$.
(2) If $(a, \gamma, k) \in (A_2)$ *and tends to* s_1 *, then* $-u_3(x, t)$ *becomes* $u_2(x, t)$.
(3) If $(a, \gamma, k) \in (A_2)$ *and tends to* s_2 *, then* $-u_3(x, t)$ *becomes* $u_4(x, t)$.
(4) If $(a, \gamma, k) \in (A_5)$ *and tends to* s_1 *, then* $-u_5(x, t)$ *becomes* $u_6(x, t)$.
(5) If $(a, \gamma, k) \in (A_6)$ *and tends to* s_1 *, then* $-u_7(x, t)$ *becomes* $u_6(x, t)$.
(6) If $(a, \gamma, k) \in (A_6)$ *and tends to* s_2 *, then* $-u_7(x, t)$ *becomes* $u_8(x, t)$.

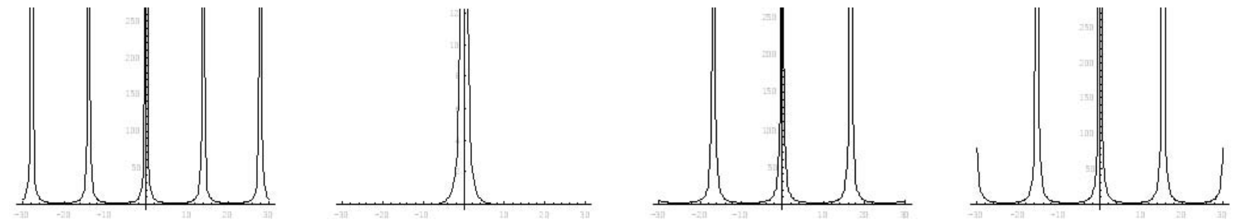
Theorem 4 *For given constant $c > 0$ and $\gamma < c$, the solution $u_i(\xi)(i = 1, \ldots, 8)$ have the following relations:*

(1) When $(a, \gamma, k) \in (B_1)$ and tends to s_1 , then $u_1(x, t)$ becomes $u_2(x, t)$.
(2) When $(a, \gamma, k) \in (B_2)$ and tends to s_1 , then $u_3(x, t)$ becomes $u_2(x, t)$.
(3) When $(a, \gamma, k) \in (B_2)$ and tends to s_2 , then $u_3(x, t)$ becomes $u_4(x, t)$.
(4) When $(a, \gamma, k) \in (B_5)$ and tends to s_1 , then $u_5(x, t)$ becomes $u_6(x, t)$.
(5) When $(a, \gamma, k) \in (B_6)$ and tends to s_1 , then $u_7(x, t)$ becomes $u_6(x, t)$.
(6)When $(a, \gamma, k) \in (B_6)$ and tends to s_2 , then $u_7(x, t)$ becomes $u_8(x, t)$.

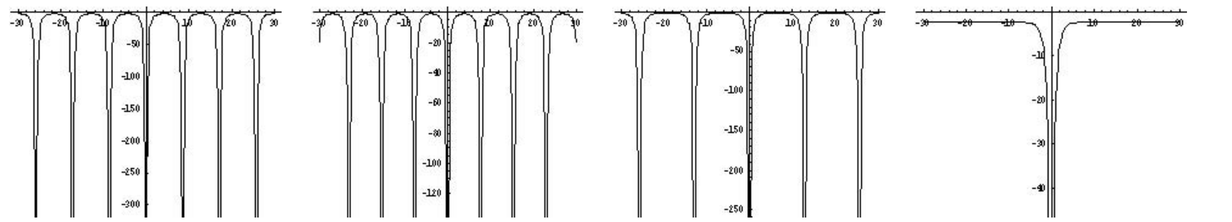
Let $\xi = x - ct$, then $u_i(x - ct)$ becomes $u_i(\xi)(i = 1, \ldots, 8)$. For given c and (a, γ, k) satisfying the conditions in Theorem 1 or 2, we can draw the graphs of $u_i(\xi)(i = 1, \ldots, 8)$

Example 1 *Letting $c = 2$, $a = 1$ and $\gamma = 1$, then from (1.7) and (1.8) it follows that $\mathrm{s}_1(1, 1) = 5/6$ and $\mathrm{s}_2(1, 1) = 11/12$. Taking $k = 1/2, 5/6, 7/8$ and $11/12$ respectively, then it is seen that$(\mathrm{a},\gamma,\mathrm{k}) = (1, 1, \frac{7}{8}) \in (\mathrm{A}_1)$, $(\mathrm{a},\gamma,\mathrm{k}) = (1, 1, \frac{5}{6}) \in \mathrm{s}_1$,$(\mathrm{a},\gamma,\mathrm{k}) = (1, 1, \frac{1}{2}) \in (\mathrm{A}_2)$, and $(\mathrm{a},\gamma,\mathrm{k}) = (1, 1, \frac{11}{12}) \in \mathrm{s}_2$. Substituting these data into the expressions of $u_i(\xi)(i = 1, 2, 3, 4)$, we draw their graphs as Fig.6 (a), (b), (c) and (d).*

Letting $c = 2$, $a = -1$ and $\gamma = 1$, similarly we get $\mathrm{s}_1(-1, 1) = 7/6$ and $\mathrm{s}_2(-1, 1) = 13/12$. Taking $k = 4/3, 7/6, 9/8$ and $13/12$ respectively, then it is seen that$(\mathrm{a},\gamma,\mathrm{k}) = (-1, 1, \frac{4}{3}) \in (\mathrm{A}_5)$, $(\mathrm{a},\gamma,\mathrm{k}) = (-1, 1, \frac{7}{6}) \in \mathrm{s}_1$, $(\mathrm{a},\gamma,\mathrm{k}) = (-1, 1, \frac{9}{8}) \in (\mathrm{A}_6)$ and $(\mathrm{a},\gamma,\mathrm{k}) = (-1, 1, \frac{13}{12}) \in \mathrm{s}_2$. Substituting these data into the expressions of $u_i(\xi)(i = 5, 6, 7, 8)$, we draw their graphs as Fig.6 (e), (f), (g) and (h). We can see visually that $u_2(\xi)$ and $u_8(\xi)$ blow-up at $\xi = 0$, and others blow-up periodically.



(a) graph of $u_1(\xi)$ for $(a, \gamma, k) = (1, 1, \frac{7}{8}) \in (A_1)$
(b) graph of $u_2(\xi)$ for $(a, \gamma, k) = (1, 1, \frac{5}{6}) \in (s_1)$
(c) graph of $u_3(\xi)$ for $(a, \gamma, k) = (1, 1, \frac{1}{2}) \in (A_2)$
(d) graph of $u_4(\xi)$ for $(a, \gamma, k) = (1, 1, \frac{11}{12}) \in (s_2)$



(e) graph of $u_5(\xi)$ for $(a, \gamma, k) = (-1, 1, \frac{4}{3}) \in (A_5)$
(f) graph of $u_6(\xi)$ for $(a, \gamma, k) = (-1, 1, \frac{7}{6}) \in (s_1)$
(g) graph of $u_7(\xi)$ for $(a, \gamma, k) = (-1, 1, \frac{9}{8}) \in (A_6)$
(h) graph of $u_8(\xi)$ for $(a, \gamma, k) = (-1, 1, \frac{13}{12}) \in (s_2)$

Figure 6: The graphs of $u_i(\xi)(i = 1, 2, ..., 8)$ when $c = 2$.

3 The proofs of main results

Substituting the expressions of $u_i(\xi)(i = 1, \ldots, 8)$ and $-u_1(\xi)$, $-u_3(\xi)$, $-u_5(\xi)$, $-u_7(\xi)$ and their parametric conditions into Eq.(1.6), it is not difficult to see that these expressions are solutions of Eq.(1.6) by using mathematical software Maple. Now we give the derivations of these expressions and show the relations among them. From (1.10) one see that α is defineed in (A_i), $(B_i)(i = 1, 2, 5, 6)$ and $s_j(j = 1, 2)$. The locations of $-\alpha$, $-c$, c and α are marked in Figs.2-5.

3.1 The proof of Theorem 1

For given $c > 0$ and $\gamma > c$, from (2.7), Figs.2 and 3, we get the expressions of some special orbits of system (2.4) and their corresponding integral equations as follows.

(1) When $(a, \gamma, k) \in (A_1)$, the orbit passing point $(\alpha, 0)$ has expression

$$y = \pm\left[a(\phi-\alpha)(\phi+c-\gamma)(\phi+\alpha)/6\right]^{1/2} \quad \text{for} \quad \phi \geq \alpha. \tag{3.1}$$

Substituting the expression into $\frac{d\phi}{y} = d\xi$ and integrating alone the orbit, we get its corresponding integral equation

$$\int_{\phi}^{+\infty} \frac{ds}{\sqrt{(s-\alpha)(s+c-\gamma)(s+\alpha)}} = \sqrt{\frac{a}{6}}\,|\xi| \quad \text{where} \quad \phi \geq \alpha \geq -c+\gamma \geq -\alpha, \tag{3.2}$$

Similarly we have:

(2) When $a > 0$, $\gamma > c$ and $(a, \gamma, k) \in s_1$, the orbit passing point $(-c+\gamma, 0)$ has expression

$$y = \pm(\phi+c-\gamma)\left[a(\phi-c+\gamma)/6\right]^{1/2} \quad \text{for} \quad \phi \geq -c+\gamma, \tag{3.3}$$

and its corresponding integral equation

$$\int_{\phi}^{+\infty} \frac{ds}{(s+c-\gamma)\sqrt{(s-c+\gamma)}} = \sqrt{\frac{a}{6}}\,|\xi|\,. \tag{3.4}$$

(3) When $(a, \gamma, k) \in (A_2)$ the orbit passing point $(-c+\gamma, 0)$ has expression

$$y = \pm\left[a(\phi+c-\gamma)(\phi-\alpha)(\phi+\alpha)/6\right]^{1/2} \quad \text{for} \quad \phi \geq -c+\gamma, \tag{3.5}$$

and its corresponding integral equation

$$\int_{\phi}^{+\infty} \frac{ds}{\sqrt{(s+c-\gamma)(s-\alpha)(s+\alpha)}} = \sqrt{\frac{a}{6}}\,|\xi| \quad \text{where} \quad \phi \geq -c+\gamma > \alpha > -\alpha. \tag{3.6}$$

(4) When $a > 0$, $\gamma > c$ and $(a, \gamma, k) \in s_2$, the orbit passing point $(-c+\gamma, 0)$ has expression

$$y = \pm\phi\left[a(\phi+c-\gamma)/6\right]^{1/2} \quad \text{for} \quad \phi \geq -c+\gamma, \tag{3.7}$$

and its corresponding integral equation

$$\int_{\phi}^{+\infty} \frac{ds}{s\sqrt{(s+c-\gamma)}} = -\sqrt{\frac{a}{6}}\,|\xi|\,. \tag{3.8}$$

(5) When $(a, \gamma, k) \in (A_5)$ the orbit passing point $(-\alpha, 0)$ has expression

$$y = \pm\left[|a|\,(\alpha-\phi)(-c+\gamma-\phi)(-\alpha-\phi)/6\right]^{1/2} \quad \text{for} \quad \phi \leq -\alpha, \tag{3.9}$$

and its corresponding integral equation

$$\int_{-\infty}^{\phi} \frac{ds}{\sqrt{(\alpha-s)(-c+\gamma-s)(-\alpha-s)}} = \sqrt{\frac{|a|}{6}}\,|\xi| \quad \text{where} \quad \phi \geq -\alpha > c-\gamma > \alpha. \tag{3.10}$$

(6) When $a < 0$, $\gamma > c$ and $(a, \gamma, k) \in s_1$, the orbit passing point $(c-\gamma, 0)$ has expression

$$y = \pm(-c+\gamma-\phi)\left[|a|\,(c-\gamma-\phi)/6\right]^{1/2} \quad \text{for} \quad \phi \leq c-\gamma, \tag{3.11}$$

and its corresponding integral equation

$$\int_{-\infty}^{\phi} \frac{ds}{(-c+\gamma-s)\sqrt{(c-\gamma-s)}} = \sqrt{\frac{|a|}{6}}\,|\xi|\,. \tag{3.12}$$

(7) When $(a,\gamma,k) \in (A_6)$ the orbit passing point $(-\alpha,0)$ has expression

$$y = \pm\left[|a|\,(-c+\gamma-\phi)(\alpha-\phi)(-\alpha-\phi)/6\right]^{1/2} \quad \text{for} \quad \phi \le -\alpha, \tag{3.13}$$

and its corresponding integral equation

$$\int_{-\infty}^{\phi} \frac{ds}{\sqrt{(-c+\gamma-s)(\alpha-s)(-\alpha-s)}} = \sqrt{\frac{|a|}{6}}\,|\xi| \quad \text{where} \quad \phi \le -\alpha < \alpha < -c+\gamma. \tag{3.14}$$

(8) When $a<0$, $\gamma>c$ and $(a,\gamma,k) \in s_2$, the orbit passing point $(0,0)$ has expression

$$y = \pm\phi\left[|a|\,(-c+\gamma-\phi)/6\right]^{1/2} \quad \text{for} \quad \phi \le 0, \tag{3.15}$$

and its corresponding integral equation

$$\int_{-\infty}^{\phi} \frac{ds}{s\sqrt{(-c+\gamma-s)}} = \sqrt{\frac{|a|}{6}}\,|\xi|\,. \tag{3.16}$$

Completing the integral in (3.2) it follows that

$$sn^{-1}\left(\sqrt{\frac{2\alpha}{\alpha+\phi}}, k_1\right) = \sqrt{\frac{a\alpha}{12}}\,|\xi| \quad \text{for} \quad a>0. \tag{3.17}$$

That is

$$\sqrt{\frac{2\alpha}{\alpha+\phi}} = sn\left(\sqrt{\frac{a\alpha}{12}}, k_1\right), \tag{3.18}$$

where

$$k_1 = \sqrt{\frac{\alpha+c-\gamma}{2\alpha}} for c-\gamma>0. \tag{3.19}$$

Solving Eq.(3.18) yields

$$\phi = \alpha\left[2sn^{-2}\left(\sqrt{\frac{a\alpha}{12}}\xi, k_1\right)-1\right]. \tag{3.20}$$

From (2.1) and $u=\phi(\xi)$, we obtain the periodic blow-up solution $u_1(x,t)$ as (2.8).

Similarly, completing the integral in (3.4), (3.6), (3.8), (3.10), (3.12), (3.14), (3.16) and solving the equations for ϕ respectively, we get $u_i(x,t)(i=2,\ldots,8)$ as (2.10), (2.11), (2.13), (2.14), (2.16),(2.17) and (2.19).These complete the derivations of Theorem 1.

3.2 The proof of Theorem 2

For given $c>0$ and $\gamma>c$, via (2.7) and Figs.5,6, we obtain the expressions of some special orbits of system (2.4) as follows.

(1) When $(a,\gamma,k) \in (B_1)$, the orbit passing point $(-\alpha,0)$ has expression

$$y = \pm\left[|a|\,(\alpha-\phi)(-c+\gamma-\phi)(-\alpha-\phi)/6\right]^{1/2} \quad \text{for} \quad \phi \le -\alpha. \tag{3.21}$$

(2) When $a<0$, $\gamma<c$ and $(a,\gamma,k) \in s_1$, the orbit passing point $(-c+\gamma,0)$ has expression

$$y = \pm(-c+\gamma-\phi)\left[|a|\,(-c+\gamma-\phi)/6\right]^{1/2} \quad \text{for} \quad \phi \le -c+\gamma. \tag{3.22}$$

(3) When $(a,\gamma,k)\in(B_2)$ the orbit passing point $(-c+\gamma,0)$ has expression

$$y=\pm\left[|a|\,(\alpha-\phi)(-\alpha-\phi)(-c+\gamma-\phi)/6\right]^{1/2}\quad\text{for}\quad \phi\le -c+\gamma. \tag{3.23}$$

(4) When $a<0,\gamma<c$ and $(a,\gamma,k)\in s_2$, the orbit passing point $(-c+\gamma,0)$ has expression

$$y=\pm\phi\left[|a|\,(-c+\gamma-\phi)/6\right]^{1/2}\quad\text{for}\quad \phi\le -c+\gamma. \tag{3.24}$$

(5) When $(a,\gamma,k)\in(B_5)$ the orbit passing point $(\alpha,0)$ has expression

$$y=\pm\left[|a|\,(\phi-\alpha)(\phi+c-\gamma)(\phi+\alpha)/6\right]^{1/2}\quad\text{for}\quad \phi\ge\alpha. \tag{3.25}$$

(6) When $a>0,\gamma<c$ and $(a,\gamma,k)\in s_1$, the orbit passing point $(c-\gamma,0)$ has expression

$$y=\pm(\phi+c-\gamma)\left[a(\phi-c+\gamma)/6\right]^{1/2}\quad\text{for}\quad \phi\ge c-\gamma. \tag{3.26}$$

(7) When $(a,\gamma,k)\in(B_6)$ the orbit passing point $(\alpha,0)$ has expression

$$y=\pm\left[a(\phi-\alpha)(\phi+\alpha)(\phi+c-\gamma)/6\right]^{1/2}\quad\text{for}\quad \phi\ge\alpha. \tag{3.27}$$

(8) When $a>0,\gamma<c$ and $(a,\gamma,k)\in s_2$, the orbit passing point $(0,0)$ has expression

$$y=\pm\phi\left[a(\phi+c-\gamma)/6\right]^{1/2}\quad\text{for}\quad \phi\ge 0. \tag{3.28}$$

Similar to the derivations of Theorem 1, using the expressions above to establish integral equations, then solving the integral equations for ϕ , we get the conclusions of Theorem.2.

3.3 The proof of Theorem 3

For give $c>0$, we have:

(1) When $(a,\gamma,k)\in(A_1)$ and tends to s_1 , from (1.7), (1.10), (1.11) , (1.12) and (2.9) it follows that

$$\alpha\to c,\beta_1\to\beta_2,k_1\to 1\quad\text{and}\quad sn(z,1)=\tanh z. \tag{3.29}$$

Via (2.8) and (3.29), one can see that $-u_1(x,t)$ becomes $u_2(x,t)$ when $(a,\gamma,k)\in(A_1)$ and tends to s_1.This completes the derivation of Theorem 3.(1).

(2) When $(a,\gamma,k)\in(A_2)$ and tends to s_1 , from (1.7), (1.10), (1.12), (1.13) and (2.12) it follows that

$$\alpha\to c,\beta_3\to\beta_2,k_2\to 1\quad\text{and}\quad sn(z,1)=\tanh z. \tag{3.30}$$

Through (2.11),(3.30), one can see that $-u_3(x,t)$ becomes $u_2(x,t)$ when $(a,\gamma,k)\in(A_2)$ and tends to s_1.

On the other hand, when $(a,\gamma,k)\in(A_2)$ and tends to s_2 , from (1.8), (1.10),(1.13), (1.14) and (2.12) it follows that

$$\alpha\to 0,\beta_3\to\beta_4,k_2\to 0\quad\text{and}\quad sn(z,0)=\sin z. \tag{3.31}$$

From (2.11) and (3.31), one can see that $-u_3(x,t)$ becomes $u_4(x,t)$ when $(a,\gamma,k)\in(A_2)$ and tends to s_2. This completes the derivation of Theorem 3.(2).

(3) When $(a,\gamma,k)\in(A_5)$ and tends to s_1 , from (1.7), (1.10), (1.11) , (1.12) and (2.15) it follows that

$$\alpha\to c,\beta_1\to\beta_2,k_3\to 0\quad\text{and}\quad sn(z,0)=\sin z. \tag{3.32}$$

Via (2.14) and (3.33), one can see that $-u_5(x,t)$ becomes $u_6(x,t)$ when $(a,\gamma,k)\in(A_3)$ and tends to s_1. This completes the derivation of Theorem 3.(3).

(4) When $(a,\gamma,k)\in(A_6)$ and tends to s_1 , from (1.7), (1.10), (1.12), (1.13) and (2.18) it follows that

$$\alpha\to c,\beta_3\to\beta_2,k_4\to 0\quad\text{and}\quad sn(z,0)=\sin z. \tag{3.33}$$

Via (2.17) and (3.33), one can see that $-u_7(x,t)$ becomes $u_6(x,t)$ when $(a,\gamma,k)\in(A_6)$ and tends to s_1.

On the other hand, when $(a, \gamma, k) \in (A_6)$ and tends to s_2, from (1.8), (1.10), (1.13), (1.14) and (2.18) it follows that

$$\alpha \to 0, \beta_3 \to \beta_4, k_4 \to 1 \quad \text{and} \quad sn(z, 1) = \tanh z. \tag{3.34}$$

Via (2.17) and (3.34), one can see that $-u_7(x,t)$ becomes $u_8(x,t)$ when $(a, \gamma, k) \in (A_6)$ and tends to s_2. This completes the derivation of Theorem 3.(4). About the relations (1)-(4) given in Theorem 4, he proof is similar to that above. Here we would not repeat it.

In this paper, we only discuss the bifurcation phase portraits and its corresponding solitary wave solutions and some periodic blow-up solutions when $c > 0$. As for $c = 0$ and $c < 0$, the results are similar to that above.

4 Conclusion

We considered the generalized Camassa-Holm equation and obtained solitary wave solutions, some periodic blow-up solutions and their limit forms which were given in Theorems 1 and 2.The expressions of these solutions are very simple and the periodic wave solutions tend to infinity on $\xi - u$ plane periodically.

From previous results (see Ref.[21,22]) and our result above, one can see that in Eq.(1.1) the effect of changing the convection term uu_x to u^2u_x and adding the term u_{xxx} causes not only the coexistence of ball-shaped solitary wave solution and peakon solution, but also the appearance of periodic blow-up solutions. We think that the generalized Camassa-Holm equation should have more complex phenomena waiting for discovering.

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References

[1] Camassa R, Holm D:An integrable shallow water equation with peaked soliton.*Phys. Rev. Lett.*71:1661-4(1993)

[2] Liu ZR, Qian TF:Peakons of the Camassa-Holm equation. *Appl. Math. Model.*26:473-80(2002)

[3] Liu ZR, Wang RQ, Jing ZJ:Peaked wave solutions of Camassa-Holm equation. *Chaos Soliton and Fractal.*19:77-92(2004)

[4] Constantin A, Escher J:Global weak solutions for a shallow water equation. *Indiana U. Math. J.*47(4):1527-45(1998)

[5] Constantin A, Molinet L:Global weak solutions for a shallow water equation. *Comm. Math. Phys.*211(1):45-61(2000)

[6] Lenells J:Travelling wave solutions of the Camassa-Holm equation. *J. Differ. Eq.*217(2):393-430(2005)

[7] Tian L, Xu G, Liu Z:The concave or convex peaked and smooth solutions of Camassa-Holm equaton. *Appl. Math. Mech.*23(5):557-67(2002)

[8] Dullin HR, Gottwald GA, Holm DD:An integrable shallow water equation with linear and nonlinear dispersion. *Phys. Rev. Lett.* 87(19):4501-4(2001)

[9] Guo BL, Liu ZR:Peaked wave solutions of CH-c equation.*Sci. China Ser. A: Math.* 46(5):696-709(2003)

[10] Guo BL, Liu ZR:Two new types of bounded waves of CH-c equation. *Sci. China Ser. A: Math.*48(12):1618-30(2005)

[11] Tang MY, Zhang WL:Four types of bounded wave solutions of CH-c equation. *Sci. China Ser. A: Math.* 50(1):132-52(2007)

[12] Liu ZR, Qian TF:Peakons and their bifurcation in a generalized Camassa-Holm equation. *Int. J. Bifurcat. Chaos.* 11(3):781-92(2001)

[13] Tian LX, Song XY:New peaked solitary wave solutions of the generalized Camassa-Holm equation. *Chaos Soliton and Fract.* 19(3):621-37(2004)

[14] Khuri SA:New ansatz for obtaining wave solutions of the generalized Camassa-Holm equation. *Chaos Soliton and Fract.* 25(3):705-10(2005)

[15] Lixin Tian, Jiuli Yin: New compacton solutions and solitary wave solutions of fully nonlinear generalized Camassa-Holm equations. *Chaos Soliton and Fract.* 20(2): 289-99(2004)

[16] Lixin Tian, Jiuli Yin: Stability of multi-compacton solutions and Backlund transformation in K(m,n,1). *Chaos Soliton and Fract.* 23(1):159-69(2005)

[17] Lixia Wang, Jiangbo Zhou and Lihong Ren: The Exact SolitaryWave Solutions for a Family of BBM Equation. *Int. J. Nonlinear Sci.* 1(1):58-64(2006)

[18] Xinghua Fan, Lixin Tian and Lihong Ren: New Compactons in Nonlinear Atomic Chain Equations with first-and-second-neighbour Interactions. *Int. J. Nonlinear Sci.* 1(2): 105-10(2006)

[19] Lin Ju. On Solution of the Dullin-Gottwald-Holm Equation. *Int. J. Nonlinear Sci.*1(1):43-8(2006)

[20] Lixin Tian, Qing Shi: Boundary Control of Viscous Dullin-Gottwald-Holm Equation. *Int. J. Nonlinear Sci.* 4(1): 67-75(2007)

[21] Wazwaz AM:Solitary wave solutions for modified forms of Degasperis-Procesi and Camassa-Holm equations. *Phys. Lett. A.* 352(6):500-4(2006)

[22] Wazwaz AM:New solitary wave solutions to the modified forms of Degasperis-Procesi and Camassa-Holm equations. *Appl. Math. Comput.*186:130-41(2007)

[23] Liu ZR, Ouyang ZY: A note on solitary waves for modified forms of Camassa-Holm and Degasperis-Procesi equations. *Phys. Lett. A.*366:377-81(2007)