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New Exact Travelling Wave Solutions for Regularized Long-wave, Phi-Four and Drinfeld-Sokolov Equations

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Abstract: In this paper, by using the solutions of an auxiliary ordinary differential equation, a direct algebraic method is described to construct the exact travelling wave solutions for RLW, PF equations and Drinfeld-Sokolov system. It is the method which can be adapted to solve nonlinear partial differential equations.

Key words: travelling wave solutions; regularized long-wave equation (RLW); Phi-Four equation (PF); Drinfeld-Sokolov system (DS)

1 Introduction

The investigation for the travelling wave solutions of nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Nonlinear phenomena appears in a wide variety of scientific applications such as plasma physics, solid state physics, optical fibers, biology, fluid dynamics and chemical kinetics. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. Because of the increased interest in the theory of solitary waves, a broad range of analytical methods was used in the analysis of these scientific models. Mathematical modeling of many physical systems leads to nonlinear ordinary or partial differential equations in various fields of physics and engineering. An effective method is required to analyze the mathematical model which provides solutions conforming to physical reality. Common analytic procedures linearize the system or assume that nonlinearities are relatively insignificant. Such assumptions, sometimes strongly, affect the solution with respect to the real physics of the phenomenon. In recent years, new exact solutions may help to find new phenomena. Thus seeking exact solutions of nonlinear ordinary or partial differential equation is great of importance. Various powerful mathematical methods are such as inverse scattering method [1], Backlund transformation [2-3], homogeneous balance method [4], tanh method [5-7], extended tanh method [8-10], sine-cosine method [11-13], pseudo spectral method [14], Jacobi elliptic method [15-16] and F-expansion method [17-19].

2 The auxiliary equation method

Let us now simply describe the auxiliary equation method. Suppose we are given nonlinear partial differential equation for $u(x,t)$ in the form:

$$P(u, u_t, u_x, u_{tt}, u_{xx}, ...) = 0 \tag{1}$$

We seek its wave solutions of the following form:

$$u = u(\xi), \quad \xi = k(x - ct) + \xi_0, \tag{2}$$

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where ξ_0 is an arbitrary constant, k and c are the wave number and wave speed, respectively. Under the transformation (2), Eq. (1) becomes an ordinary differential equation as

$$Q(u, u', u'', u''', ...) = 0 \tag{3}$$

By virtue of the above method we assume that the solution of Eq. (3) is of the form

$$u(\xi) = \sum_{i=0}^{m} a_i F^i(\xi), \tag{4}$$

and $F(\xi)$ is the solution of the auxiliary ordinary differential equation

$$F'^2(\xi) = q_2 F^2(\xi) + q_3 F^3(\xi) + q_4 F^4(\xi), \tag{5}$$

where q_2, q_3, q_4 are real parameters, and hence holds for $F(\xi)$,

$$\begin{cases} F'F'' = q_2 F F' + \frac{3}{2} q_3 F^2 F' + 2q_4 F^3 F', \\ F'' = q_2 F + \frac{3}{2} q_3 F^2 + 2q_4 F^3, \\ F''' = q_2 F' + 3q_3 F F' + 6q_4 F^2 F'. \end{cases} \tag{6}$$

Integer m in (4) can be determined by considering homogeneous balance between the nonlinear terms and the highest derivatives of $u(\xi)$ in Eq. (3).

In this paper, we present new types solitary wave solutions of Eq. (5), we shall seek the explicitly solitary wave solutions of some nonlinear evolution equations by using the following new solitary wave solutions of Eq. (5):

$$F(\xi) = -\sqrt{\frac{q_2}{4q_4}} \left(1 \pm \frac{\sinh(\sqrt{q_2}\xi)}{\cosh(\sqrt{q_2}\xi) \pm 1} \right), \quad q_2, q_4 > 0,\ q_3^2 - 4q_2 q_4 = 0, \tag{7}$$

and [20]

$$F(\xi) = \begin{cases} \dfrac{-q_2 q_3 \sec h^2(\frac{\sqrt{q_2}}{2}\xi)}{q_3^2 - q_2 q_4 \left(1 - \tanh(\frac{\sqrt{q_2}}{2}\xi)\right)^2}, q_2 > 0, \\ \\ \dfrac{2q_2 \sec h(\sqrt{q_2}\xi)}{\sqrt{q_3^2 - 4q_2 q_4} - q_3 \sec h(\sqrt{q_2}\xi)},\ q_3^2 - 4q_2 q_4 > 0, q_2 > 0. \end{cases} \tag{8}$$

And then by using some significant special solutions of the auxiliary ordinary differential equation, some famous equations are investigated and exact solutions are explicitly obtained with the aid of symbolic computation [17].

As a result, we obtain new explicit solitary wave solutions of some nonlinear evolution equations.

3 Exact travelling wave solutions of RLW equation

The RLW equation is given by

$$u_t + a u_x - 6 u u_x - b u_{xxt} = 0 \quad a, b > 0, \tag{9}$$

where a, b are real constants [11]. Making the transformation $u(x, t) = u(\xi), \xi = k(x - ct) + \xi_0$, Eq. (9) becomes

$$(a - c)u' - 6uu' + bck^2 u''' = 0. \tag{10}$$

Following the balancing act procedure we balance the highest order of derivative term u''' with the highest power nonlinear term uu', yields $m = 2$. Therefore we may choose the solution of Eq. (10) in the form

$$u(\xi) = a_0 + a_1 F + a_2 F^2, \tag{11}$$

where a_0, a_1 and a_2 are constants to be determined later. It is easy to deduce that

$$u' = a_1F' + 2a_2FF', \tag{12}$$

$$u'' = 6a_2q_4F^4 + (2a_1q_4 + 5a_2q_3)F^3 + (\frac{3}{2}a_1q_3 + 4a_2q_2)F^2 + a_1q_2F, \tag{13}$$

$$u''' = 24a_2q_4F^3F' + 3(2a_1q_4 + 5a_2q_3)F^2F' + 2(\frac{3}{2}a_1q_3 + 4a_2q_2)FF' + a_1q_2F', \tag{14}$$

$$uu' = a_0a_1F' + (2a_0a_2 + a_1^2)FF' + 3a_1a_2F^2F' + 2a_2^2F^3F'. \tag{15}$$

Substituting (12)-(15) into (10), setting each coefficient of F^iF' (i = 0, 1, 2, 3) to zero , yields a set of equations for a_i,

$$(a-c)a_1 - 6a_0a_1 + bck^2a_1q_2 = 0, \tag{16}$$

$$2(a-c)a_2 - 6(2a_0a_2 + a_1^2) + bck^2(3a_1q_3 + 8a_2q_4) = 0, \tag{17}$$

$$-18a_1a_2 + 3bck^2(2a_1q_4 + 5a_2q_3) = 0, \tag{18}$$

$$-12a_2^2 + 24bck^2a_2q_4 = 0. \tag{19}$$

We obtain

$$a_0 = \frac{a-c+bck^2q_2}{6}, \quad a_1 = bck^2q_3, \quad a_2 = 2bck^2q_4, \quad 16q_4^2 = 4q_2q_4 + 3q_3^2. \tag{20}$$

Substituting (20) into (11) with (7) and (8), we obtain new solutions of Eq. (9)

$$u_1(\xi) = \frac{a-c+bck^2q_2}{6} - bck^2\sqrt{\frac{4q_4-q_2}{3}}\left(1 \pm \frac{\sinh(\sqrt{q_2}\xi)}{\cosh(\sqrt{q_2}\xi) \pm 1}\right) + 2bck^2q_4\left(1 \pm \frac{\sinh(\sqrt{q_2}\xi)}{\cosh(\sqrt{q_2}\xi) \pm 1}\right)^2$$

where $q_2, q_4 > 0$, and

$$u_2(\xi) = \frac{a-c+bck^2q_2}{6} - \frac{bck^2q_2q_3^2\sec h^2(\frac{\sqrt{q_2}}{2}\xi)}{q_3^2 - q_2q_4(1-\tanh(\frac{\sqrt{q_2}}{2}\xi)} + 2bck^2q_4\left(\frac{-q_2q_3\sec h^2(\frac{\sqrt{q_2}}{2}\xi)}{q_3^2 - q_2q_4\left(1-\tanh(\frac{\sqrt{q_2}}{2}\xi)\right)^2}\right)^2$$

where $q_2 > 0$, and

$$u_3(\xi) = \frac{a-c+bck^2q_2}{6} - \frac{2bck^2q_2\sec h(\sqrt{q_2}\xi)}{\sqrt{q_3^2-4q_2q_4} - q_3\sec h(\sqrt{q_2}\xi)} + 2bck^2q_4\left(\frac{2q_2\sec h(\sqrt{q_2}\xi)}{\sqrt{q_3^2-4q_2q_4} - q_3\sec h(\sqrt{q_2}\xi)}\right)^2$$

where $q_3^2 - 4q_2q_4 > 0, q_2 > 0$.

4 Exact travelling wave solutions of PF equation

We consider the Phi-four equation

$$u_{tt} - au_{xx} - u + u^3 = 0, \quad a > 0, \tag{21}$$

where a is real constant [11]. After making transformation $u(x,t) = u(\xi), \xi = k(x-ct) + \xi_0$, Eq. (21) becomes

$$k^2(c^2-a)u'' - u - u^3 = 0, \tag{22}$$

Using the method mentioned above, we balance the highest order of derivative term u'' with the highest power nonlinear term u^3, yields $m = 1$. Therefore we may choose the solution of Eq. (22) in the form

$$u(\xi) = a_0 + a_1F, \tag{23}$$

where a_0 and a_1 are constants to be determined later. It is easy to deduce that

$$u' = a_1 F', \tag{24}$$

$$u'' = a_1 F'' = a_1(q_2 F + \frac{3}{2} q_3 F^2 + 2q_4 F^3), \tag{25}$$

Substituting (24)-(25) into (22), setting each coefficient of F^i (i = 0, 1, 2, 3) to zero , yields a set of equations for a_0, a_1,

$$-a_0 + a_0^3 = 0, \tag{26}$$

$$k^2(c^2 - a)a_1 q_2 - a_1 + 3a_0^2 a_1 = 0, \tag{27}$$

$$\frac{3}{2} k^2(c^2 - a)a_1 q_3 + 3a_1^2 a_0 = 0, \tag{28}$$

$$2k^2(c^2 - a)a_1 q_4 + a_1^3 = 0. \tag{29}$$

We obtain

Case A:

$$a_0 = \pm 1, \quad a_1 = \frac{q_3}{q_2}, \quad k^2(c^2 - a)q_2 + 2 = 0. \tag{30}$$

Case B:

$$a_0 = \pm 1, \quad a_1 = \pm 2\sqrt{\frac{q_4}{q_2}}, \quad k^2(c^2 - a)q_2 + 2 = 0. \tag{31}$$

Substituting (30) and (31) into (23) with (7) and (8), we obtain new solutions of Eq. (21)

Case A:

$$u_1(\xi) = \pm 1 - \left(1 \pm \frac{\sinh(\sqrt{q_2}\xi)}{\cosh(\sqrt{q_2}\xi) \pm 1}\right)$$

where $q_2 > 0,\ q_3^2 - 4q_2 q_4 = 0$, and

$$u_2(\xi) = \pm 1 - \frac{q_3^2 \sec h^2(\frac{\sqrt{q_2}}{2}\xi)}{q_3^2 - q_2 q_4 \left(1 - \tanh(\frac{\sqrt{q_2}}{2}\xi)\right)^2}$$

where $q_2 > 0$, and

$$u_3(\xi) = \pm 1 + \frac{2q_3 \sec h(\sqrt{q_2}\xi)}{\sqrt{q_3^2 - 4q_2 q_4} - q_3 \sec h(\sqrt{q_2}\xi)}$$

where $q_3^2 - 4q_2 q_4 > 0, q_2 > 0$.

Case B:

$$u_1(\xi) = \pm 1 \pm \left(1 \pm \frac{\sinh(\sqrt{q_2}\xi)}{\cosh(\sqrt{q_2}\xi) \pm 1}\right)$$

where $q_2 > 0$, and

$$u_2(\xi) = \pm 1 \pm 2\frac{\sqrt{q_2 q_4} q_3 \sec h^2(\frac{\sqrt{q_2}}{2}\xi)}{q_3^2 - q_2 q_4 \left(1 - \tanh(\frac{\sqrt{q_2}}{2}\xi)\right)^2} \tag{32}$$

where $q_2, q_4 > 0$, and

$$u_3(\xi) = \pm 1 \pm 4\frac{\sqrt{q_2 q_4} \sec h(\sqrt{q_2}\xi)}{\sqrt{q_3^2 - 4q_2 q_4} - q_3 \sec h(\sqrt{q_2}\xi)}$$

where $q_3^2 - 4q_2 q_4 > 0, q_2, q_4 > 0$.

5 Exact travelling wave solutions of DS system

We next consider the Drinfeld-Sokolov system

$$\begin{aligned} &u_t + (v^2)_x = 0, \\ &v_t - av_{xxx} + 3bu_xv + 3duv_x = 0. \end{aligned} \tag{33}$$

where a, b and d are real constants [21]. Using the wave variable $\xi = k(x - ct) + \xi_0$, the system (33) is carried to a system of ODEs

$$\begin{aligned} &-cku' + (v^2)' = 0, \\ &cv' + ak^2v''' - 3bvu' - 3duv' = 0. \end{aligned} \tag{34}$$

Integrating the first equation in the system and neglecting the constant of integration we find

$$cku = v^2. \tag{35}$$

Substituting (35) into the second equation of the system and integrating we find

$$c^2kv + ack^3v'' - (2b + d)v^3 = 0. \tag{36}$$

Balancing v'' with v^3 in (34) gives

$$\begin{aligned} &n + 2 = 3n, \\ &m = 2n. \end{aligned} \tag{37}$$

so that

$$\begin{aligned} &m = 2, \\ &n = 1. \end{aligned} \tag{38}$$

The auxiliary equation method (4) admits the solution of Eq. (33) in the form

$$v(\xi) = b_0 + b_1F, \tag{39}$$

where b_0 and b_1 are constants to be determined later. It is easy to deduce that

$$v' = b_1F', \tag{40}$$

$$v'' = b_1F'' = b_1(q_2F + \frac{3}{2}q_3F^2 + 2q_4F^3), \tag{41}$$

Substituting (40)-(41) into (36), setting each coefficient of F^i (i = 0, 1, 2, 3) to zero , yields a set of equations for b_0, b_1,

$$kc^2b_0 - (2b + d)b_0^3 = 0, \tag{42}$$

$$kc^2b_1 - 3(2b + d)b_1b_0^2 + ack^3b_1q_2 = 0, \tag{43}$$

$$\frac{3}{2}ack^3b_1q_3 - 3(2b + d)b_1^2b_0 = 0, \tag{44}$$

$$2ack^3b_1q_4 - (2b + d)b_1^3 = 0. \tag{45}$$

We obtain

$$b_0 = \pm\frac{c\sqrt{k}}{\sqrt{2b + d}}, \quad b_1 = \pm\frac{k\sqrt{2ackq_4}}{\sqrt{2b + d}}, \quad q_3^2 = 4q_2q_4. \tag{46}$$

Substituting (46) into (36) with (35) and (39), we obtain new solutions of Eq. (33)

$$u_1(\xi) = \pm\frac{c}{2b + d}\left(1 \pm \frac{\sinh(\sqrt{q_2}\xi)}{\cosh(\sqrt{q_2}\xi) \pm 1}\right)^2,$$

$$v_1(\xi) = \pm\frac{c\sqrt{k}}{\sqrt{2b + d}}\left(1 \pm \frac{\sinh(\sqrt{q_2}\xi)}{\cosh(\sqrt{q_2}\xi) \pm 1}\right).$$

where $q_2 > 0$. $q_3^2 = 4q_2q_4$. Due to this there is no solution providing Eq. (8).

6 Conclusion

The auxiliary equation method was succesfully used to establish solitary wave solutions. Many well known nonlinear wave equations were handled by this method to show the new solutions compared to the solutions obtained in [12],[21]. The performance of the auxiliary equation method is reliable and effective and gives more solutions. The applied method will be used in further works to establish more entirely new solutions for other kinds of nonlinear wave equations. The availability of computer systems like *Mathematica* or *Maple* facilitates the tedious algebraic calculations. The method which we have proposed in this letter is also a standard, direct and computerizable method, which allows us to do complicated and tedious algebraic calculation.

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