Box Dimensions of a Class of Self-conformal Sets

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Abstract: In this paper, we construct a class of self-conformal sets based on self-similar sets, and obtain the formula for their box dimensions supported by "finite" type of self-similar sets.

Keywords: self-conformal set; box dimension; finite type

1 Introduction

Let \( \{ \phi_i \}_{i=1}^q \) be an iterated function system of contractive similitudes on \( \mathbb{R} \), defined by
\[
\phi_i(x) = a_i x + c_i, \quad 1 \leq i \leq q, \tag{1}
\]
and \( \{ \phi'_i \}_{i=1}^q \) be a contractive self-conformal IFS on \( \mathbb{R} \), defined by
\[
\phi'_i(x) = a_i x + b_i x^2 + c_i, \quad 1 \leq i \leq q, \tag{2}
\]
where for all \( i \), \( 0 < a_i < 1, \ 0 < b_i < 1, \ 0 < a_i + b_i < 1, \ c_i \in \mathbb{R} \). Let \( F \) be self-similar set (or attractor) defined by the IFS \( \{ \phi_i \}_{i=1}^q \), and \( F' \) be the self-conformal set defined by the IFS \( \{ \phi'_i \}_{i=1}^q \). It is well known that if the contractive self-similar IFS \( \{ \phi_i \}_{i=1}^q \) satisfies the open set condition (OSC), then \( \dim_B(F) = s \), where \( s \) is the unique solution of
\[
\sum_{i=1}^q a_i^s = 1. \tag{3}
\]
It has been the subject of several studies[1-5]. However, in the absence of the OSC the images of \( F \) under the \( \phi_i \) have overlaps and the above dimension formula fails in general. In this case it is much harder to compute dimension of \( F \). Ngai and Wang[6,7] introduced the notion of finite type and described a scheme for the computation of dimension when the finite type occurs. The scheme based on the notion of finite type can be outlined as follows.

Let \( \sum_q = \{1, 2, ..., q\} \) and \( \sum_q^* = \bigcup_{n \geq 0} \sum_q^n \) be the set of all finite words in \( \sum_q \), where \( \sum_q^n \) is the set of all words of length \( n \), with \( \sum_q^0 \) containing only the empty word \( \emptyset \). For \( j \in \sum_q^n \) let \( |j| = n \) denote the length of \( j \). For \( i \in \sum_q \) and \( j \in \sum_q^n \) let \( ij \in \sum_q^{n+m} \) be the concatenation of \( i \) and \( j \), and call \( i \) an initial segment of \( ij \).

Let \( j = (j_1, j_2, \cdots, j_m) \in \sum_q^m \), then
\[
\phi_j = \phi_{j_1} \circ \cdots \circ \phi_{j_m}, \quad a_j = a_{j_1} \cdots a_{j_m}. \tag{4}
\]

Now let \( \rho = \min \{ a_i \} \). For all \( k \geq 0 \) define
\[
\Lambda_k = \{ j \in \sum_q^m | a_j \leq \rho^k \text{ but } a_i > \rho^k \text{ if } i \text{ is a proper initial segment of } j \}. \tag{5}
\]
Intuitively, all \( \phi_j \) for \( j \in \Lambda_k \) have comparable contraction ratios, which are in the order of \( \rho^k \).
Let $\mathcal{V} = \bigcup_{k \geq 0} \mathcal{V}_k$, where $\mathcal{V}_0 = \{(I, 0)\}$ and $\mathcal{V}_k = \{(\phi_i, k) : i \in \Lambda_k\}$ for all integer $k \geq 1$. A directed graph $\mathcal{G}$ is constructed with $\mathcal{V}$ as the set of vertices and $j \in \sum_q^*$ as the directed edge connecting from $(\phi_i, k)$ to $(\phi_j, k + 1)$ (this implies that $i \in \Lambda_k$, $(\phi_i, k) \in \mathcal{V}_k$, and $ij \in \Lambda_{k+1}$, $(\phi_j, k + 1) \in \mathcal{V}_{k+1}$). The vertex $(\phi_j, k + 1)$ is called an offspring of the vertex $(\phi_i, k)$, and conversely, the vertex $(\phi_i, k)$ is called the parent of the vertex $(\phi_j, k + 1)$. Thus each vertex in $\mathcal{V}_k$ has at least one offspring in $\mathcal{V}_{k+1}$, and each vertex in $\mathcal{V}_{k+1}$ has at least one parent in $\mathcal{V}_k$. The reduced graph $G_k$ is obtained from $\mathcal{G}$ by first removing all but the smallest (in the lexicographical order) directed edge going to a vertex. The set of the $k$-th order vertices of the reduced graph $G_k$ is denoted by $\mathcal{V}_{R,k}$ and $\mathcal{V}_R = \bigcup_{k \geq 0} \mathcal{V}_{R,k}$.

Fix any nonempty bounded open set $\Omega \subset \mathbb{R}$ which is invariant under $\{\phi_i\}_{i=1}^q$, i.e., $\bigcup_{i=1}^q \phi_i(\Omega) \subset \Omega$. Two vertices $v, v' \in \mathcal{V}_k$ (allowing $v = v'$) are neighbors (with respect to $\Omega$) if $\phi_i(\Omega) \cap \phi_j'(\Omega) \neq \emptyset$ (where for $v = (\phi_i, k) \in \mathcal{V}_k$ we use the convenient notation $\phi_v := \phi_i$). The set of vertices

$$\Omega(v) = \{v' : v' \text{ is a neighbor of } v\}$$

is called the neighborhood of $v$ (with respect to $\Omega$). Two vertices $v \in \mathcal{V}_k$ and $v' \in \mathcal{V}_k'$ are equivalent, denoted by $v \sim v'$, if, for $\tau := \phi_{i'} \circ \phi_i^{-1} : \mathbb{R} \to \mathbb{R}$, the following conditions are satisfied[7]:

1. $\{\phi_{i'} : u \in \Omega(v)\} = \{\tau \circ \phi_u : u \in \Omega(v)\}$;
2. for $u \in \Omega(v)$ and $u' \in \Omega(v')$ such that $\phi_{i'} = \tau \circ \phi_u$, and for any positive integer $l \geq 1$, an index $i \in \sum_q^*$ satisfies $(\phi_i \circ \phi_i, k + l) \in \mathcal{V}_{k+l}$ if and only if it satisfies $(\phi_{i'} \circ \phi_i, k + l) \in \mathcal{V}_{k'+l}$.

The IFS $\{\phi_i\}_{i=1}^q$ is said to be of finite type if there are finitely many distinct neighborhood types.

In this case we can define the incidence matrix $S = [s_{ij}]$ for IFS. Suppose that there are $N$ neighborhood types. Choose any vertex $v$ has neighborhood type $i$. Its offspring in some reduced graph will have various neighborhood types. The entry $s_{ij}$ denotes the number of offspring that have neighborhood type $j$.

**Theorem 1** (See[6]) Let $\{\phi_i\}_{i=1}^q$ be an iterated function system defined as in (1). Suppose that the iterated function system is of finite type with respect to a bounded invariant open set $\Omega$, and let $S$ be the corresponding incidence matrix. Then the attractor $F$ of the iterated function system satisfies

$$\dim_{H}(F) = \dim_{B}(F) = \frac{\log \lambda}{-\log \rho},$$

where $\rho = \min_{i} a_i$ and $\lambda = \lambda(S)$ is the spectral radius of $S$.

In this paper, for the contractive self-conformal IFS $\{\phi_i^*\}_{i=1}^q$, we define

$$\Lambda'_{k} = \{j \in \sum_q^* \mid a_j \leq \rho^k \text{ but } a_j > \rho^k \text{ if } i \text{ is a proper initial segment of } j\},$$

where $\sum_q^*$ defined as $\sum_q^*$. We denote $\mathcal{V}_0 = \{(I, 0)\}$ and $\mathcal{V}_k = \{(\phi_i^*, k) : i \in \Lambda'_{k}\}$ for all $k \geq 1$. This leads to our main result.

**Condition 2** For $\{\phi_i^*\}_{i=1}^q$ satisfies: let $E, U$ be any subsets in $\mathbb{R}$ with $\text{diam}(U) \leq K_1 \rho^k$ and $\text{diam}(E) \leq K_2$. Then there exists an $M = M(K_1, K_2) > 0$, such that for all $k \geq 0$, \#$\{v \in \mathcal{V}_k' \mid U \cap \phi_i^*(E) \neq \emptyset\} \leq M$.

**Theorem 3** Let $\{\phi_i\}_{i=1}^q, \{\phi_i^*\}_{i=1}^q$ be iterated function systems defined as (1), (2). Suppose that $\{\phi_i\}_{i=1}^q$ is of finite type with respect to a bounded invariant open set $\Omega$. $\{\phi_i^*\}_{i=1}^q$ satisfies Condition 2 and $|\mathcal{V}_k'| = |\mathcal{V}_k|$. Then the self-conformal set $F'$ satisfies

$$\dim_B(F') = \frac{\log \lambda}{-\log \rho},$$

where $\rho = \min_{i} a_i$, $S$ be the corresponding incidence matrix for $\{\phi_i\}_{i=1}^q$, and $\lambda = \lambda(S)$ is the spectral radius of $S$.

**Corollary 4** If $\{\phi_i\}_{i=1}^q$ and $\{\phi_i^*\}_{i=1}^q$ are both of no complete overlap, we will obtain

$$\dim_B(F') = \frac{\log \lambda}{-\log \rho},$$

where $\rho = \min_{i} a_i$ and $\lambda = \lambda(S)$ is the spectral radius of $S$. 

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2 The proof of the results

**Lemma 5** Let $|\phi_i(\Omega)|$ denote the diameter of $\phi_i(\Omega)$, then there exists a positive real number $P > 1$ such that for any $i \in \sum^*$,

$$1 < \frac{|\phi_i(\Omega)|}{|\phi_i(\Omega)|} < P,$$

where $\Omega$ is a bounded invariant open set of $\{\phi_i\}_{i=1}^q$ and $\{\phi_i\}_{i=1}^q$.

**Proof.** Let

$$f(n) = \prod_{i=1}^n \frac{1}{1 + AB^i|\Omega|},$$

where

$$A = \max_{1 \leq i \leq q} \left\{ \frac{b_i}{a_i} \right\}, \quad B = \max_{1 \leq i \leq q} \{a_i + b_i\} < 1.$$

Obviously, $0 < f(n) < 1$, $(n = 1, 2, \cdots)$ and $\{f(n)\}$ is monotone decreasing. So we have

$$0 \leq \sigma < 1,$$

where $\sigma := \prod_{i=1}^\infty \frac{1}{1 + AB^i|\Omega|}$.

Let

$$g(n) = \prod_{i=1}^n \frac{1}{1 + \frac{i}{\pi^2}} > 0,$$

$\{g(n)\}$ is also monotone decreasing. Note that

$$\prod_{i=1}^\infty \frac{1}{1 + \frac{i}{\pi^2}} = \prod_{i=1}^\infty \frac{i^2}{i^2 + 1} \geq \frac{1}{2} \prod_{i=2}^\infty \frac{i^2 - 1}{i^2 + 1} = \frac{\pi}{2} \cosh(\pi) > 0.$$

There exists a positive integer $N$ large enough, we have

$$\prod_{i=N}^\infty \frac{1}{1 + AB^i|\Omega|} > \prod_{i=N}^\infty \frac{1}{1 + \frac{i}{\pi^2}}.$$

By (3), we have

$$\prod_{i=N}^\infty \frac{1}{1 + \frac{i}{\pi^2}} > 0, \text{ so } \sigma > 0.$$

Thus we can put

$$P = \left\lfloor \frac{1}{\sigma} \right\rfloor + 1.$$

For any positive integer number $k$, and any $i \in \Sigma^k_q$, we have

$$\frac{|\phi_i(\Omega)|}{|\phi_i(\Omega)|} = \prod_{j=1}^k \frac{a_{i_j}}{a_{i_j} + b_{i_j} |\phi_{i_1} \circ \cdots \circ \phi_{i_{j-1}}(\Omega)|} = \prod_{j=1}^k \frac{1}{1 + \frac{b_{i_j}}{a_{i_j}} |\phi_{i_1} \circ \cdots \circ \phi_{i_{j-1}}(\Omega)|}.$$

From the definition of $\{\phi_i\}_{i=1}^q$, we can obtain that

$$\prod_{i=1}^\infty \frac{1}{1 + AB^i|\Omega|} < \frac{|\phi_i(\Omega)|}{|\phi_i(\Omega)|} < 1.$$

Therefore

$$1 < \frac{|\phi_i(\Omega)|}{|\phi_i(\Omega)|} < P, \text{ for any } i \in \Sigma^*.$$

\[\blacksquare\]
Lemma 6 Let $\{\phi_i\}_{i=1}^q, \{\phi'_i\}_{i=1}^q$ be iterated function systems defined as (1), (2). Suppose that $\{\phi_i\}_{i=1}^q$ is of finite type with respect to a bounded invariant open set $\Omega$. $\{\phi'_i\}_{i=1}^q$ satisfies Condition 2. Then the self-conformal set $F'$ satisfies

$$
\liminf_{k \to \infty} \frac{\log |V'_k|}{-k \log \rho} \leq \dim_B(F') \leq \dim_B(F') \leq \limsup_{k \to \infty} \frac{\log |V'_k|}{-k \log \rho}.
$$

Proof. By Lemma 5, let $N(cP\rho^k)$ be the minimal number of balls of radius $cP\rho^k$ needed to cover $F'$. Then for any $c_1, c_2 > 0$ there exist positive constants $K^+(c_1, c_2)$ and $K^-(c_1, c_2)$ such that

$$
K^-(c_1, c_2) N(c_1 P \rho^k) \leq N(c_2 P \rho^k) \leq K^+(c_1, c_2) N(c_1 P \rho^k).
$$

Observe that $F' = \bigcup_{i \in A'_k} \phi'_i(F') = \bigcup_{v \in V'_k} \phi'_v(F')$, and there exists a $c_0 > 0$ such that each $\phi'_v(F')$ can be covered by a ball of radius $c_0 P \rho^k$. So we have

$$
|V'_k| \geq N(c_0 P \rho^k).
$$

Now let $B_1, \ldots, B_N(\delta)$ be balls of radius $\delta > 0$ that cover $F'$. We may uniquely write $\delta = cP\rho^k$ for some $k$ and $\rho < c \leq 1$. By Condition 2, the cardinality of $\{v \in V'_k | B_j \cap \phi'_v(F') \neq \emptyset\}$ is bounded by some fixed $M > 0$ for all $1 \leq j \leq N(\delta)$. Therefore $|V'_k| \leq MN(\delta)$. On other hand,

$$
|V'_k| \geq N(c_0 P \rho^k) \geq K^-(c, c_0) N(cP \rho^k) = K^-(c, c_0) N(\delta).
$$

Therefore

$$
\overline{\dim}_B(F') = \liminf_{\delta \to 0} \frac{\log N(\delta)}{-\log \delta} \leq \liminf_{k \to \infty} \frac{\log(|V'_k|/M)}{-\log(cP\rho^k)} = \liminf_{k \to \infty} \frac{\log |V'_k|}{-k \log \rho}.
$$

Similarly, applying (4) we obtain

$$
\underline{\dim}_B(F') = \limsup_{\delta \to 0} \frac{\log N(\delta)}{-\log \delta} \leq \limsup_{k \to \infty} \frac{\log(|V'_k|/K^-(c, c_0))}{-\log(cP\rho^k)} = \limsup_{k \to \infty} \frac{\log |V'_k|}{-k \log \rho}.
$$

So

$$
\liminf_{k \to \infty} \frac{\log |V'_k|}{-k \log \rho} \leq \overline{\dim}_B(F') \leq \underline{\dim}_B(F') \leq \limsup_{k \to \infty} \frac{\log |V'_k|}{-k \log \rho}.
$$

Proof of Theorem 3.

Proof. We can evaluate $|V_k|$ using the incidence matrix $S$ of the finite type IFS $\{\phi_i\}_{i=1}^q$ with respect to $\Omega$. By[6]

$$
|V_k| = e_1^T S^k \xi,
$$

where $\xi = [1, 1, \ldots, 1]^T$ and $e_1 = [1, 0, \ldots, 0]^T$ are vectors in $\mathbb{R}^N$. Then

$$
\lim_{k \to \infty} (e_1^T S^k \xi)^{1/k} = \lambda.
$$

Now let $\varepsilon > 0$ be arbitrary. Then there exists $K$ large enough, for all $k \geq K$,

$$
(\lambda - \varepsilon)^k < e_1^T S^k \xi < (\lambda + \varepsilon)^k.
$$

Using the fact that $|V_k| = e_1^T S^k \xi, |V'_k| = |V_k|$, and applying Lemma 6, we get

$$
\frac{\log (\lambda - \varepsilon)}{-\log \rho} \leq \liminf_{k \to \infty} \frac{\log |V_k|}{-k \log \rho} \leq \overline{\dim}_B(F') \leq \underline{\dim}_B(F') \leq \limsup_{k \to \infty} \frac{\log |V_k|}{-k \log \rho} \leq \frac{\log (\lambda + \varepsilon)}{-\log \rho}.
$$

Letting $\varepsilon \to 0$ yields

$$
\dim_B(F') = \frac{\log \lambda}{-\log \rho}.
$$
Proof of Corollary 4.

Proof. When \( \{\phi_i\}_{i=1}^q \) and \( \{\phi_i'\}_{i=1}^q \) are also of no complete overlap, it is easy to obtain

\[
|V_k'| = |V_k|,
\]

so

\[
\dim_B(F') = \frac{\log \lambda}{-\log \rho}.
\]

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References


