An Extended Riccati Equation Rational Expansion Method and its Applications

M A Abdou 1,2 *

1 Theoretical Research Group, Physics Department, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt
2 Faculty of Education for Girls, Physics Department, King Kahlid University, Bisha, Saudia Arabia

(Received 1 February 2008, accepted 12 March 2008)

Abstract: Based on computerized symbolic computation and a new general ansatz, an extended Riccati equation rational expansion method is presented to construct multiple exact solutions for nonlinear evolution equations and implemented in a computer algebraic system. The validity and reliability of the method are tested by its application to four nonlinear evolution equations arising in physics, namely, generalized nonlinear Schrodinger equation with a source term, Hirota-Satsuma coupled KdV system, coupled Maccaris equations and generalized-Zakharov equations. As a result, we obtain several new kinds of exact solutions. It is shown that the extended rational expansion method provides a very effective and powerful mathematical tool for solving other nonlinear evolution equations arising in physics.

Key words: Symbolic computation method; Generalized rational expansion method; Nonlinear evolution equations; New exact travelling wave solutions

1 Introduction

The application of computer algebra to science has a bright future. In the field of nonlinear science, to find as many and general as possible exact solutions for a nonlinear system is one of the most fundamental and significant study. In the line with the development of computerized symbolic computation, much work has been focused on the various extensions and application of the known algebraic methods to construct the solutions of nonlinear evolution equations [1 – 33].

On the lines of the rational expansion thought, we propose a new algebraic method, named, Riccati equation rational expansion method [9, 10]. Although the new method does not recover all the known rational expansion method, it can obtain some new exact solutions. For illustration, we apply the extended method to solve the generalized nonlinear Schrodinger equation with a source term, Hirota-Satsuma coupled KdV system, coupled Maccaris equations and generalized-Zakharov equations, and successfully construct new and more general solutions, including rational formal hyperbolic function solutions, rational formal triangular periodic solutions, and rational solutions. Here, we present an effective extension to the tanh method and develop a new extended algebraic method to uniformly construct a series of travelling wave solutions including soliton, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions for general nonlinear equations.

This paper is organized as follows, in Section 2, we describe briefly the new generalized extended rational expansion method. In Section 3, we apply the proposed method to solve four nonlinear evolution equations of special interest in nonlinear mathematical physics. In Section 4, conclusions will be presented in final.

*Corresponding author. E-mail address: m_abdou_eg@yahoo.com

Copyright © World Academic Press, World Academic Union
IJNS.2009.02.15/200
2 Extended Riccati equation rational expansion method

In the following, we would like to outline the main steps of our method: Step 1. For given nonlinear evolution equations:

$$\psi(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, \ldots) = 0$$  \hspace{1cm} (1)

The main steps of our proposed method are given as follows:

By using the travelling wave transformation $u(x, t) = u(\xi), \xi = kx + ct$, then Eq.(1) reduces to:

$$\psi(u, u', ku', k^2u'', \ldots) = 0$$  \hspace{1cm} (2)

Step 2: We introduce a new ansatz in terms of finite rational formal expansion as follows:

$$u = a_0 + \sum_{j=1}^{N} \frac{a_j \phi(\xi) + b_j \phi^{j-1}(\xi) \phi'(\xi)}{\mu \phi(\xi) + 1)^j}$$  \hspace{1cm} (3)

and the new variable $\phi(\xi)$ satisfying:

$$\phi'(\xi) = \sqrt{h_0 + h_1 \phi(\xi) + h_2 \phi(\xi)^2 + h_3 \phi(\xi)^3 + h_4 \phi(\xi)^4}$$  \hspace{1cm} (4)

The parameter $N$ can be found by balancing the highest order derivative term and the nonlinear terms in (2).

Step 3. Substitute Eq.(3) into Eq.(2) along with Eq.(4), setting the coefficients of all powers of $\phi^i(\xi)\phi'(\xi)(i = 0, 1, \ldots)$ to zero, we get a system of over-determined system of algebraic equations. Solving the over-determined system of nonlinear algebraic equations by use of Maple, we would end up with the explicit expressions for $a_0, a_j$ and $b_j$.

Step 4. Inserting the values $a_0, a_j$ and $b_j$ obtained in Step(3) back into Eq.(3) then solving it, we may get its solutions of Eq.(1).

Step 5. It is well known that the general solutions of Eq.(4) are:

Case A. Eq.(4) admits two kinds of polynomials solutions as follows:

$$\phi_1 = \sqrt{h_0} \xi, \quad h_1 = h_2 = h_3 = h_4 = 0, h_0 > 0$$
$$\phi_2 = -\frac{h_0}{h_1} + \frac{1}{4} h_1 \xi^2, \quad h_2 = h_3 = h_4 = 0, h_1 \neq 0$$  \hspace{1cm} (5)

Case B. Eq.(4) admits two kinds of exponential solutions, namely:

$$\phi_3 = -\frac{h_1}{2h_2} + \exp(\sqrt{h_2} \xi), \quad h_3 = h_4 = 0, h_0 = \frac{h_2^2}{4h_3}, h_2 > 0$$
$$\phi_4 = \frac{h_3}{2h_4} \exp(\frac{h_3}{2\sqrt{-h_4}} \xi), \quad h_0 = h_1 = h_2 = 0, h_4 > 0$$  \hspace{1cm} (6)

Case C. Eq.(4) admits two kinds of rational, namely:

$$\phi_5 = -\frac{1}{\sqrt{h_4}} \xi, \quad h_0 = h_1 = h_2 = h_3 = 0, h_4 > 0$$
$$\phi_6 = -\frac{4h_3}{h_3^2 \xi^2 - 4h_4}, \quad h_0 = h_1 = h_2 = 0$$  \hspace{1cm} (7)

Case D. Eq.(4) admits the triangular solutions as follows:

$$\phi_7 = -\frac{h_1}{2h_2} + \frac{h_1}{2h_2} \sin(\sqrt{-h_2} \xi), \quad h_0 = h_3 = h_4 = 0, h_2 < 0,$$
$$\phi_8 = \frac{-h_0}{h_2} \sin(\sqrt{-h_2} \xi), \quad h_1 = h_3 = h_4 = 0, h_2 > 0, h_0 < 0,$$

IJNS email for contribution: editor@nonlinearscience.org.uk
\[ \phi_0 = \sqrt{-\frac{h_2}{h_4}} \sec(\sqrt{-h_2} \xi), \ h_0 = h_3 = 0, h_2 < 0, h_4 > 0, \]

\[ \phi_{10} = \frac{h_2}{h_3} \sec^2\left(\frac{\sqrt{-h_2}}{2} \xi\right), \ h_0 = h_1 = h_4 = 0, h_2 < 0, \]

\[ \phi_{11} = \frac{h_2}{2h_4} \tan\left(\frac{\sqrt{-h_2}}{2} \xi\right), h_1 = h_3 = 0, h_0 = \frac{h_2^2}{4h_4}, h_2 > 0, h_4 > 0 \]  

(8)

Case E. Eq.(4) admits the kinds of hyperbolic solutions, namely:

\[ \phi_{12} = -\frac{h_1}{2h_2} + \frac{h_3}{2h_2} \sinh(2\sqrt{h_2} \xi), h_0 = h_3 = h_4 = 0, h_2 > 0, \]

\[ \phi_{13} = \sqrt{\frac{h_0}{h_2}} \sinh(\sqrt{h_2} \xi), h_1 = h_3 = 0, h_2 > 0, h_4 > 0, \]

\[ \phi_{14} = \sqrt{\frac{h_2}{h_4}} \sech(\sqrt{h_2} \xi), h_0 = h_1 = h_3 = 0, h_2 > 0, h_4 < 0, \]

\[ \phi_{15} = -\frac{h_3}{h_4} \sech^2\left(\frac{\sqrt{h_2}}{2} \xi\right), h_0 = h_1 = h_4 = 0, h_2 > 0, \]

\[ \phi_{16} = \sqrt{-\frac{h_2}{2h_4}} \tanh\left(\frac{\sqrt{-h_2}}{2} \xi\right), h_1 = h_3 = 0, h_0 = \frac{h_2^2}{4h_4}, h_2 < 0, h_4 > 0 \]  

(9)

Case F. Eq.(4) admits three Jacobi elliptic function solutions as follows:

\[ \phi_{17} = \sqrt{-\frac{h_2 m^2}{h_4 (2m^2 - 1)}} \cn\left(\sqrt{\frac{h_2}{(2m^2 - 1)}} \xi\right), h_4 < 0, h_2 > 0, h_1 = h_3 = 0, h_0 = \frac{m^2 h_2^2 (m^2 - 1)}{h_4 (2m^2 - 1)^2} \]

\[ \phi_{18} = \sqrt{-\frac{h_2 m^2}{h_4 (m^2 - 1)}} \sn\left(-\sqrt{\frac{h_2}{(m^2 + 1)}} \xi\right), h_4 > 0, h_2 < 0, h_1 = h_3 = 0, h_0 = \frac{h_2 m^2}{h_4 (m^2 + 1)^2} \]

\[ \phi_{19} = \sqrt{-\frac{h_2}{h_4 (2 - m^2)}} \dn\left(\sqrt{\frac{h_2}{(2 - m^2)}} \xi\right), h_4 < 0, h_2 > 0, h_0 = \frac{h_2^2 (1 - m^2)}{h_4 (m^2 - 2)^2}, h_1 = h_3 = 0 \]  

(10)

As \( m \to 1 \), the Jacobi elliptic periodic solutions degenerates to hyperbolic functions, i.e. \( \sn(\xi) = \tanh(\xi), \cn(\xi) = \sech(\xi) \) and \( \dn(\xi) = \sech(\xi) \). When \( m \to 0 \), the Jacobi elliptic periodic solutions degenerates to triangular functions, i.e. \( \sn(\xi) = \sin(\xi), \cn(\xi) = \cos(\xi) \) and \( \dn(\xi) = 1 \).

Case G. Eq.(4) admits a Weierstrass elliptic function solution:

\[ \phi_{20} = \psi\left(\frac{\sqrt{h_3}}{2} \xi, g_2, g_3\right), h_2 = h_4 = 0, h_3 > 0 \]  

(11)

where \( g_2 = -\frac{4h_1}{h_3} \) and \( g_3 = -\frac{4h_0}{h_3} \).

3 New Applications

In order to illustrate the effectiveness and convenience of the method, we consider four nonlinear evolution equations arising in physics, namely, generalized nonlinear Schrodinger equation with a source term, Hirota-Satsuma coupled KdV system, coupled Maccari’s equations and generalized-Zakharov equations.

3.1 Example(1). Generalized nonlinear (GNLS) Schrodinger equation

Let us first consider the GNLS equation [13]:

\[ iu_t + au_{xx} + bu|u|^2 + icu_{xxx} + id(u|u|^2)_x = ke^{i(x(t) - wt)}, \]

(12)

where \( \xi = \alpha(x - vt) \) is a real function and \( a, b, c, d, k, \alpha, c, v, w \) are all real. The GNLS Eq.(12) plays an important role in many nonlinear science. It arises as an asymptotic limit of a slowly varying dispersive wave envelope.
Eq.(16) can be expressed by:

$$u(x, t) = \phi(\xi)e^{i[\xi(\xi)-wt]} \quad (13)$$

For convenience, let $\chi = \beta \xi + x_0$, where $\beta$ and $x_0$ are real constants. Then, separating the real and imaginary parts of Eq.(12), we obtain

$$c\alpha^3 \phi''' + (-\alpha v + 2a\beta \alpha^2 - 3c\alpha^3 \beta^3) \phi' + 3d\alpha \phi^2 \phi' = 0, \quad (14)$$

$$\left( a\alpha^2 - 3c\phi^3 \beta \right) \phi'' + (\alpha \beta v + w - a\alpha^2 \beta^2 + c\alpha^3 \beta^3) \phi + (b - d\alpha \beta) \phi^3 - k = 0 \quad (15)$$

Integrating Eq.(14) w.r.t. $\xi$ once yields:

$$A\phi'''(\xi) + B\phi(\xi) + d\phi^3(\xi) - C = 0, \quad (16)$$

where $A = c\alpha^2, B = -v + 2a\alpha \beta - 3c\alpha^3 \beta^2, C$ is an integration constant. Since the same function $\phi(\xi)$ satisfies two Eqs.(16) and (15), we obtain the following constraint conditions:

$$\frac{(a\alpha^2 - 3c\phi^3 \beta)}{A} = \frac{(\alpha \beta v + w - a\alpha^2 \beta^2 + c\alpha^3 \beta^3)}{B} = \frac{(b - d\alpha \beta)}{d} = \frac{k}{C} \quad (17)$$

By virtue of the technique of solution Eq.(16) using the rational expansion method. Considering the homogenous balance between $\phi'''(\xi)$ and $\phi^3(\xi)$ in Eq.(16), yields $N = 1$, we suppose that the solution of Eq.(16) can be expressed by:

$$\phi(\xi) = a_0 + \frac{a_1 \phi(\xi) + b_1 \phi'(\xi)}{\mu \phi(\xi) + 1}, \quad (18)$$

where $a_0, a_1$, and $b_1$ are constants to be determined later. Substituting Eq.(18) along with (4) into Eq.(16), equating the coefficients of all powers of $\phi'(\xi) \phi(\xi)$, we get a set of over determined algebraic system for $a_0, a_1, b_1, d, C, \mu$ and $b_1$. Solving the system of over-determined algebraic equations using Wu-elimination method [34], we get four cases of the solutions as follows:

**Case[1]**: When $h_3 = h_4 = 0$:

$$C = 0, a_0 = a_0, b_1 = b_1, \mu = \mu, a_1 = -a_0 \mu, h_0 = -\frac{2A\mu^2 a_0^2 - Bb_1^2}{4a_0 b_1 A\mu^2}, h_1 = \frac{b_1 B}{2A\mu^2 A^3}$$

$$d = -\frac{2A\mu^2}{b_1^2}, h_2 = \frac{Bb_1}{4a_0}, \quad (19)$$

**Case[2]**: When $h_1 = h_3 = 0$:

$$a_1 = 0, d = -\frac{2A\mu^2}{b_1^2}, C = -\frac{a_0 (2A\mu^2 a_0^2 - Bb_1^2)}{b_1^2}, a_0 = a_0, \mu = \mu, b_1 = b_1, h_4 = 0$$

$$h_0 = -\frac{-Bb_1^2 + 6A\mu^2 a_0^2}{Ab_1 \mu^2 a_0}, h_2 = \frac{a_0^2}{b_1} \quad (20)$$

**Case[3]**: When $h_4 = 0$:

$$a_0 = \frac{\sqrt{8} \sqrt{\frac{B}{A}} b_1}{8 \mu}, a_1 = 0, b_1 = b_1, \mu = \mu, C = 0, h_0 = h_0, d = -\frac{8A\mu^2}{b_1^2}$$

$$h_1 = \frac{\left( \frac{3}{4} A\mu \sqrt{8} \sqrt{\frac{B}{A}} h_0 + B \right) \sqrt{8}}{2A \sqrt{\frac{B}{A}}},$$

**IJNS email for contribution:** editor@nonlinearscience.org.uk
\[ h_2 = \frac{3 \left( \frac{1}{2} A\mu \sqrt{8} \sqrt{\frac{2}{\chi} h_0 + B} \right) \mu \sqrt{8}}{4A \sqrt{\frac{2}{\chi}}}, \quad h_3 = \frac{\mu^2 \left( \frac{1}{2} A\mu \sqrt{8} \sqrt{\frac{2}{\chi} h_0 + B} \right) \sqrt{8}}{4A \sqrt{\frac{2}{\chi}}}, \quad (21) \]

Case[4]: When \( h_1 = h_2 = h_3 = h_4 \neq 0 \):

\[ a_1 = 0, b_1 = b_1, \mu = \mu, C = 0, d = -\frac{18A\mu^2}{b_1^2}, h_1, a_0 = \frac{b_1(h_1 - \frac{A}{A} \sqrt{\sqrt{BA}})}{6\mu}, \]

\[ h_4 = \frac{4\mu^2}{3}(h_1 - \frac{A}{A} \sqrt{\sqrt{BA}}), \quad h_2 = 2\mu(h_1 - \frac{A}{A} \sqrt{\sqrt{BA}}), \]

\[ h_3 = \frac{\mu^3}{3}(h_1 - \frac{A}{A} \sqrt{\sqrt{BA}}), \quad h_0 = \frac{A}{A} \sqrt{\sqrt{BA}}, \quad (22) \]

According Case[1], inserting Eq.(19) into Eqs.(18) and (13), admits to the exact travelling wave solution of Eq.(12) as follows:

\[ u_1(x, t) = \left[ a_0 + \frac{-a_0\mu\phi(\xi) + b_1\phi'(\xi)}{\mu\phi(\xi) + 1} \right] e^{i[\chi(\xi) - wt]}, \quad (23) \]

\[ \phi(\xi) = \sqrt{h_0} \xi, \quad h_1 = h_2 = 0, h_0 > 0; \phi(\xi) = -\frac{h_0}{h_1^2} + \frac{1}{4} h_1^2, h_2 = h_1 \neq 0, \]

\[ \phi(\xi) = -\frac{h_1}{2h_2} + \exp(\sqrt{h_2} \xi), h_0 = \frac{h_1^2}{4h_2}, h_2 > 0, \]

\[ \phi(\xi) = -\frac{h_1}{2h_2} + \frac{h_1}{2h_2} \sinh(\sqrt{h_2} \xi), h_0 = 0, h_2 < 0, \]

\[ \phi(\xi) = -\frac{h_1}{2h_2} + \frac{h_1}{2h_2} \sin(\sqrt{h_2} \xi), h_0 = 0, h_2 > 0, \]

where \( \chi = \beta \xi + x_0 \) and \( \xi = \alpha(x - vt) \).

From Eq.(21) and using Eqs.(18) and (13), we have the exact travelling wave solutions of Eq.(12) in the following from:

\[ u_2(x, t) = \left[ \sqrt{8} \sqrt{\frac{b_1}{8\mu}} + \frac{b_1\phi'(\xi)}{(\mu\phi(\xi) + 1)} \right] e^{i[\chi(\xi) - wt]}, \quad (24) \]

\[ \phi(\xi) = -\frac{h_2}{h_3} \text{sech}{\left( \frac{\sqrt{h_2} \xi}{2} \right)}, h_0 = h_1 = 0, h_2 > 0, \]

\[ \phi(\xi) = -\frac{h_2}{h_3} \text{sech}{\left( \frac{\sqrt{h_2} \xi}{2} \right)}, h_0 = h_1 = 0, h_2 < 0, \]

\[ \phi(\xi) = \frac{A}{h_3 \xi^2}, h_0 = h_1 = h_2 = 0; \phi(\xi) = \psi(\sqrt{h_3} \xi, g_2, g_3), h_2 = 0, h_3 > 0, \]

where \( g_2 = -\frac{2h_2}{h_3} \) and \( g_3 = -\frac{4h_0}{h_3} \).

With the aid of Eq.(22) and Eqs.(18) and (13), we have:

\[ u_3(x, t) = \left[ \frac{b_1(h_1 - \frac{A}{A} \sqrt{\sqrt{BA}})}{6\mu} + \frac{b_1\phi'(\xi)}{(\mu\phi(\xi) + 1)} \right] e^{i[\chi(\xi) - wt]}, \quad (25) \]

\[ \phi(\xi) = \sqrt{\frac{h_2}{h_4}} \text{sech}{\left( \sqrt{h_2} \xi \right)}, h_0 = 0, h_2 > 0, h_4 < 0, \]

\[ \phi(\xi) = \sqrt{\frac{h_2}{2h_4}} \text{tanh}{\left( \sqrt{-h_2 / 2} \xi \right)}, h_0 = \frac{h_3^2}{4h_4}, h_2 < 0, h_4 > 0, \]

\[ \phi(\xi) = \sqrt{\frac{-h_2}{h_4}} \text{sec}{\left( \sqrt{-h_2} \xi \right)}, h_0 = 0, h_2 < 0, h_4 > 0, \]

IINS homepage: http://www.nonlinearscience.org.uk/
\[ \phi(\xi) = \sqrt{\frac{h_2}{2h_4}} \tan(\sqrt{\frac{h_2}{2}}\xi), h_0 = \frac{h_2^2}{4h_4}, h_2 > 0, h_4 > 0, \]
\[ \phi(\xi) = -\frac{1}{\sqrt{h_4}} \xi, h_0 = h_2 = 0, h_4 > 0 \]
\[ \phi(\xi) = \sin(\xi), h_0 = 1, h_2 = -(m^2 + 1), h_4 = m^2, \]
\[ \phi(\xi) = \cos(\xi), h_0 = 1 - m^2, h_2 = 2m^2 - 1, h_4 = -m^2, \]
\[ \phi(\xi) = \tan(\xi), h_0 = m^2 - 1, h_2 = 2 - m^2 - 1, h_4 = -1, \]
\[ \phi(\xi) = \csc(\xi), h_0 = 0, h_2 = 1 - m^2, h_4 = -1, \]
\[ \phi(\xi) = \sec(\xi), h_0 = 1, h_2 = 2 - m^2, h_4 = 1 - m^2, \]
\[ \phi(\xi) = \cot(\xi) \pm ds(\xi), h_0 = \frac{m^2}{4}, h_2 = \frac{m^2 - 2}{2}, h_4 = \frac{1}{4} \]

By means of Eq.(20) with the aid of Eqs.(18) and (13), admits to the exact travelling wave solutions of Eq.(12) as follows:

\[ u_4(x, t) = [a_0 + \frac{b_1 \phi(\xi)}{\mu \phi(\xi) + 1}]e^{i[x(\xi) - wt]} \quad (26) \]

3.2. Example(2). Generalized Hirota-Satsuma coupled KdV system

A second instructive model is the Hirota-Satsuma coupled KdV system [5]:

\[ u_t = \frac{1}{4} u_{xxx} + 3uu_x + 3(w - v^2)_x; v_t = -\frac{1}{2} v_{xxx} - 3uv_x, \]
\[ w_t = -\frac{1}{2} w_{xxx} - 3uw_x \quad (27) \]

When \( w = 0 \), Eqs.(27) reduces to be the well-known Hirota-Satsuma coupled KdV system. We seek travelling wave solutions for Eqs.(27) as:

\[ u(x, t) = u(\xi), v(x, t) = v(\xi), w(x, t) = w(\xi), \xi = k(x - ct) \quad (28) \]

Substituting Eq.(28) into (27) yields

\[ -cku'' = \frac{1}{4} k^3 u''' + 3ku' + 3k(w - v^2)' \quad (29) \]
\[ -ckv'' = \frac{1}{2} k^3 v''' - 3kw' \quad (30) \]
\[ -ckw'' = \frac{1}{2} k^3 w''' - 3kw' \quad (31) \]

Setting

\[ u = \alpha v^2 + \beta_0 v + \gamma, w = A_0 v + B_0 \quad (32) \]

where \( \alpha, \gamma, \beta_0, A_0 \) and \( B_0 \) are constants. Inserting Eq.(32) into (30) and (31) integrating once we know that (30) and (31) give rise to the same equation:

\[ k^2 v'' = -2\alpha v^3 - 3\beta_0 v^2 + 2(c - 3\gamma)v + k_1, \quad (33) \]

where \( k_1 \) is an integration constant. Integrating (33) once again we have:

\[ k^2 v' = -\alpha v^4 - 2\beta_0 v^3 + 2(c - 3\gamma)v^2 + k_1 v + k_2, \quad (34) \]

where \( k_2 \) is an integration constant. By means of Eqs.(32-34) we get:

\[ k^2 u'' = 2\alpha k^2 v' + k^2 (2\alpha v + \beta_0)v'' = 2\alpha [-\alpha v^4 - 2\beta_0 v^3] \]

IJNS email for contribution: editor@nonlinearscience.org.uk
\[ +2(c - 3\gamma)v^2 + 2k_1v + k_2 + (2\alpha v + \beta_0)(-2\alpha v^3 - 3\beta_0 v^2 + 2(c - 3\gamma)v + k_1) \]  

Integrating (29) once we have:

\[ \frac{1}{4}k^2 u'' + \frac{3}{2}u^2 + cu + 3(w - v^2) + k_3 = 0, \]  

where \( k_3 \) is an integration constant. Inserting (32) and (35) into (36) gives:

\[ 3\alpha c - 3\alpha \gamma + 3\beta_0^2 - 3 = 0; \frac{1}{2}(\alpha k_1 + \beta_0 c + \gamma \beta_0) + A_0 = 0, \]  

\[ \frac{1}{4}(2\alpha k_2 + \beta_0 k_1) + \frac{3}{2}\gamma^2 + c\gamma + 3B_0 + k_3 = 0 \]  

Setting

\[ k_1 = \frac{1}{2\alpha^2} [\beta_0^3 + 2\alpha \beta_0^2 - 6\alpha \beta_0 \gamma], \]  

\[ v(\xi) = aP(\xi) - \frac{\beta_0}{2\alpha} \]  

Therefore from Eq.(37), we have

\[ AP''(\xi) + BP(\xi) + 2\alpha a^3 P^3(\xi) = 0 \]  

In the same manner, Eq.(39) coincides with Eq.(16) with \( C = 0 \), where \( A \) and \( B \) are defined by:

\[ A = k^2, B = -a\left(\frac{3\beta_0^2}{2\alpha} + 2c - 6\gamma\right) \]  

A series of exact travelling wave solutions are obtained for Eq.(39) in the similar way of Eq.(16). We omit this discussion here for simplicity.

### 3.3. Example(3). The coupled Maccaris equations

The coupled Maccaris equations \([35]\) reads:

\[ iQ_t + Q_{xx} + QR = 0, \]  

\[ R_t + R_y + (|Q|^2)_x = 0. \]  

In order to seek exact solutions of Eqs.(40), we suppose:

\[ Q(x, y, t) = u(x, y, t) \exp[i(kx + \alpha y + \lambda t + l)], \]  

where \( k, \alpha \) and \( \lambda \) are constants to be determined later, \( l \) is an arbitrary constant. Substituting Eq.(41) into Eqs.(40) and yields PDEs:

\[ i(u_t + 2ku_x) + u_{xx} - (\lambda + k^2)u + uR = 0, \]  

\[ R_t + R_y + (u^2)_x = 0 \]  

Using the transformation

\[ u = u(\xi), R = R(\xi), \xi = w(x + \beta y - 2kt + x_0), \]  

where \( \beta \) and \( w \) are constants to be determined later, \( x_0 \) is an arbitrary constant, Eqs.(42) become:

\[ w^2 u - (\lambda + k^2)u + uR = 0, \]  

\[ (\beta - 2k)R' + (u^2)' = 0, \]  

where prime denotes the differential with respect to \( \xi \). Integrating Eq.(45) with respect to \( \xi \) and taking the integration constant as zero yields:

\[ R = -\frac{1}{\beta - 2k} u^2(\xi) \]  

IJNS homepage: http://www.nonlinearscience.org.uk/
Substituting Eq.(46) into (44), yields:

\[ Au''(\xi) + B u(\xi) - \frac{1}{\beta - 2k} u^{3}(\xi) = 0 \] (47)

Eq.(47) coincides with Eq.(16) with \( C = 0 \), where \( A \) and \( B \) are defined by:

\[ A = \beta^{2}, \quad B = -(\lambda + k^{2}) \]

A new exact travelling wave solution are obtained for Eq.(47) in the same manner of Eq.(16). We omit this discussion here for simplicity.

3.4. Example(4). The generalized-Zakharov equations

In this case, the generalized Zakharov equations for the complex envelope [33] reads:

\[ i\psi_{t} + \psi_{xx} - 2\lambda|\psi|^{2}\psi + 2\psi v = 0, \]
\[ v_{tt} - v_{xx} + (|\psi|^{2})_{xx} = 0, \] (48)

Let us assume the travelling wave solution of Eqs.(48) in the form:

\[ \psi(x, t) = e^{iu(\xi)}, \quad v = v(\xi), \]
\[ \eta = \alpha x + \beta t, \quad \xi = k(x - 2\alpha t), \] (49)

where \( u(\xi) \) and \( v(\xi) \) are real functions, the constants \( \alpha, \beta \) and \( k \) are to be determined. Substituting (49) into Eqs.(48), we have:

\[ k^{2}u'' + 2uv - (\alpha^{2} + \beta)u - 2\lambda u^{3} = 0, \] (50)
\[ k^{2}(4\alpha^{2} - 1)v'' + k^{2}(u^{2})'' = 0 \] (51)

In order to simplify ODEs (50) and (51), integrating Eq.(51) once and taking integration constant to zero, and integrating yields:

\[ v(\xi) = \frac{u^{2}}{(1 - 4\alpha^{2})} + C_{0}, \quad if \alpha \neq \frac{1}{4}. \] (52)

where \( C_{0} \)-integration constant. Inserting Eq.(52) into (50), we have:

\[ Au'' + Bu + 2\frac{1}{1 - 4\alpha^{2} - \lambda}|u|^{3} = 0 \] (53)

As proceeding before, Eq.(53) coincides with Eq.(16) in case of \( C = 0 \), where \( A \) and \( B \) are defined by:

\[ A = k^{2}, \quad B = [2C - \alpha^{2} - \beta] \]

It is to be noted that a series of new exact travelling wave solution are obtained for Eq.(53) as Eq.(16). We omit this discussion here for simplicity.

4 Conclusion

An extended rational expansion method with a computerized symbolic computation has been proposed to obtain the new exact solutions to four nonlinear evolution equations arising in nonlinear mathematical physics. The validity of this method has been tested by applying it successfully to generalized nonlinear Schrodinger equations, generalized Hirota-Satsuma coupled KdV system, coupled Maccari equations and generalized-Zakharov equations. The underlying mechanism for a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions act to change wave forms in many nonlinear equations arising in physics.

It is worthwhile to mention that the method is straightforward and concise, and it can also be applied to other nonlinear evolution equations in nonlinear physical problems. This is our task in the future.
Acknowledgements

The author would like to express his great thankful to Prof. S.A. El-Wakil for his encouragement and supervision and to the referees for their useful comments and discussions.

References


*IINS homepage:* http://www.nonlinearscience.org.uk/


*IJNS email for contribution: editor@nonlinearscience.org.uk*