Existence of Solutions to a Non-autonomous p-Laplacian Equation

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Abstract: We consider solutions to the boundary value problem

\[
\begin{cases}
-\Delta_p u = \lambda f(x, u), & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
\]

where \(\Delta_p\) denotes the p-Laplacian operator defined by \(\Delta_p z = \text{div}(|\nabla z|^{p-2}\nabla z)\); \(p > 1, \lambda > 0, \Omega\) is a smooth bounded domain in \(R^N (N \geq 1)\) with smooth boundary \(\partial \Omega\), and \(f\) is a smooth function such that

\[
\lim_{u \to +\infty} \frac{f(x, u)}{u^{p-1}} = \zeta,
\]

uniformly in \(x\),

\[
f(x, -u) = -f(x, u),
\]

for \(x \in \Omega\), and \(u > 0\) and

\[
\frac{\partial f}{\partial u}(x, u) \geq 0.
\]

We denote by \(\lambda_1\) the first eigenvalue of

\[
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\]
\begin{align}
\begin{cases}
\Delta_p \phi + \lambda |\phi|^{p-2} \phi = 0, & x \in \Omega, \\
\phi(x) = 0, & x \in \partial \Omega,
\end{cases}
\end{align}

with positive principal eigenfunction $\phi_1$ satisfying $\|\phi_1\|_\infty = 1$ (see [5]).

Let $I = [a, b] \subset (\lambda_1, \lambda_1 + \delta)$, where $\delta > 0$ is the constant in the anti-maximum principle (see [5]), and for $\lambda \in I$ consider the unique solution $z_\lambda$ of the boundary value problem

\begin{align}
\begin{cases}
\Delta_p z_\lambda + \lambda z_\lambda^{p-1} = 1, & x \in \Omega, \\
z_\lambda = 0, & x \in \partial \Omega.
\end{cases}
\end{align}

We assume that

$$
\mu_1 = \inf_{x \in \Omega, \lambda \in I} z_\lambda(x), \quad \mu_2 = \sup_{x \in \Omega, \lambda \in I} z_\lambda(x),
$$

and

$$
\mu = \max\{|\mu_1|, |\mu_2|\}.
$$

We also strengthen the assumptions on function $f$. Namely, we assume that there exists $m > 0$ such that

$$
y^{p-1} + m > f(x, y) > y^{p-1} - m,
$$

for all $y \in [-(bm)^{1-p} \mu, (bm)^{1-p} \mu]$. Now assume that

$$
\zeta < \frac{1}{b\|w_\alpha\|_\infty^{p-1}},
$$

where $w_\alpha$ is the unique positive solution to

\begin{align}
\begin{cases}
-\Delta_p w_\alpha = 1, & x \in \Omega, \\
w_\alpha = 0, & x \in \partial \Omega.
\end{cases}
\end{align}

In this paper we discuss the existence of three solutions where one of them is positive, while another is negative, for $\lambda$ near $\lambda_1$, and for $\lambda$ large when $f$ is p-sublinear.

Equation (1) has been studied by many authors in the last years (see [3, 4, 6, 8, 10, 12]). These problems are interesting in applications (combustion, mathematical biology, chemical reactions) and raise many difficult mathematical problems. An existence result using variational method for sublinear problems was given in [2]. See [1], where the authors studied two point boundary value problems of the form

\begin{align}
\begin{cases}
-u'' = \lambda f(x, u), & x \in (-1, 1), \\
u(-1) = 0 = u(1),
\end{cases}
\end{align}

and obtained the existence results via the sub and supersolutions method. The purpose of this paper is to extend some of this study to the p-Laplacian case and higher dimensions.

## 2 Main results

We start with some preparation. Let $\Omega \subset R^N$ be a smooth bounded domain and $1 < p < N$. Let $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ denote the usual sobolev spaces.

**Definition 2.1.** We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution to (1) if for any $w \in W_0^{1,p}(\Omega)$ we have

$$
\int_\Omega |\nabla u|^{p-2} \nabla u. \nabla w dx = \int_\Omega \lambda f(x, u) w dx.
$$

From the standard regularity results for equation (1), the weak solutions belong to the function class $C^{1,\alpha}(\overline{\Omega})$ for $\alpha \in (0, 1)$ (see [5]).
Definition 2.2. We say that $\psi \in W^{1,p}_0(\Omega) (\phi \in W^{1,p}_0(\Omega))$ is a subsolution (a supersolution) to (1) if

$$\int_{\Omega} |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla w dx - \int_{\Omega} \lambda f(x, \psi) w dx \leq 0 \quad (\geq 0),$$

holds for all $w \in W^{1,p}_0(\Omega)$ such that $w \geq 0$ for $x \in \Omega$ and $\psi \leq 0 \leq \phi$ for $x \in \partial \Omega$. Then by Weak comparison principle, (see [5]) equation (1) has a $C^1(\Omega)$ solution $u$ such that $\psi \leq u \leq \phi$.

We prove multiplicity by a sub-super solution result for the $p$-Laplacian case discussed in [9]. This result extends the corresponding result for the Laplacian ($p = 2$) case proved in [11]. The result is as follows:

Lemma 2.3. Suppose there exist a subsolution $\psi_1$, a strict supersolution $\phi_1$, a strict subsolution $\psi_2$ and a supersolution $\phi_2$ for (1) such that $\psi_1 < \phi_1 < \phi_2$, $\psi_1 < \psi_2 < \phi_2$ and $\psi_2 \not< \phi_1$. Then (1) has at least three distinct solutions $u_i (i = 1, 2, 3)$ such that $\psi_1 \leq u_1 < u_2 < u_3 \leq \phi_2$.

Now, we recall the anti-maximum principle (see [5]) in the following form: Let $\lambda_1$ is as defined before. Then there exists a $\delta(\Omega) > 0$ such that the solution $z_\lambda$ of (6) for $\lambda \in (\lambda_1, \lambda_1 + \delta)$, is positive for $x \in \Omega$ and is such that $\frac{\partial z_\lambda}{\partial n} < 0$, for $x \in \partial \Omega$.

Our main existence result is the following theorem:

Theorem 2.4. Let $I = [a, b] \subset (\lambda_1, \lambda_1 + \delta)$ and (2) – (4), (7) – (8) hold. Then there exist at least three solutions to (1) for $\lambda \in I$, where one of them is a positive solution, while another is a negative solution.

Proof. Let $v_1(x) = (bm)^{\frac{1}{p-1}} z_\lambda(x)$, $u_2(x) = -v_1(x)$ and $w \in W^{1,p}_0(\Omega)$; such that $w \geq 0$ for $x \in \Omega$. Then

$$\int_{\Omega} |\nabla v_1|^{p-2} \nabla v_1 \cdot \nabla w dx = \int_{\Omega} (-\Delta_p v_1) w dx = \int_{\Omega} bm(-\Delta_p z_\lambda) w dx$$

$$= \int_{\Omega} bm(\lambda z_\lambda^{p-1} - 1) w dx \leq \int_{\Omega} bm(\lambda z_\lambda^{p-1} - \frac{\lambda}{b}) w dx$$

$$= \int_{\Omega} \lambda(bm z_\lambda^{p-1} - m) w dx = \int_{\Omega} \lambda(v_1^{p-1} - m) w < \int_{\Omega} \lambda f(x, v_1) w dx.$$  

Here in the last inequality, we use the fact that $f(x, y) > y^{p-1} - m$. Similarly, we have

$$\int_{\Omega} |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla w dx = \int_{\Omega} (\Delta_p v_1) w dx = \int_{\Omega} bm(\Delta_p z_\lambda) w dx$$

$$= \int_{\Omega} bm(1 - \lambda z_\lambda^{p-1}) w dx$$

$$\geq \int_{\Omega} bm(\frac{\lambda}{b} - \lambda z_\lambda^{p-1}) w dx$$

$$= \int_{\Omega} \lambda(m - bm z_\lambda^{p-1}) w dx$$

$$= \int_{\Omega} \lambda(u_2^{p-1} + m) w > \int_{\Omega} \lambda f(x, u_2) w dx.$$  

Here in the last inequality, we use the fact that $f(x, y) < y^{p-1} + m$. Thus $v_1$ is a strict subsolution, while $u_2$ is a strict supersolution. Note that since $\lambda \in (\lambda_1, \lambda_1 + \delta)$, then $z_\lambda > 0$, for $x \in \Omega$. Then $v_1 > 0; x \in \Omega$, while $u_2 < 0; x \in \Omega$. Now let $u_1(x) = J^{\frac{1}{p-1}} w_\alpha(x), v_2 = -J^{\frac{1}{p-1}} w_\alpha(x)$ where $J > 0$ is large enough so that
Clearly, here we do not need that 

\[
\frac{1}{\lambda \| w_\alpha \|_\infty^{p-1}} \geq \frac{1}{b \| w_\alpha \|_\infty^{p-1}} \geq \frac{f(x, J \frac{1}{p-1} \| w_\alpha \|_\infty)}{(J \frac{1}{p-1} \| w_\alpha \|_\infty)^{p-1}}
\]

and

\[
u_1 \geq v_1, u_1 \geq u_2, v_2 \leq v_1, v_2 \leq u_2.
\]

Here (10) is possible by (2), (8) and (11) is possible by the maximum principle. Then

\[
\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla w dx = \int_\Omega (-\Delta_p u_1) w dx = \int_\Omega J w dx \\
\geq \int_\Omega b f(x, J \frac{1}{p-1} \| w_\alpha \|_\infty) w dx \\
\geq \int_\Omega \lambda f(x, J \frac{1}{p-1} \| w_\alpha \|_\infty) w dx \\
\geq \int_\Omega \lambda f(x, J \frac{1}{p-1} w_\alpha) w dx = \int_\Omega \lambda f(x, u_1) w dx;
\]

where \(x \in \Omega\) and \(w_\alpha\) is as defined before. Here in the last inequality, we use the fact that \(\frac{\partial f}{\partial u}(x, u) \geq 0\). Similarly we have

\[
\int_\Omega |\nabla v_2|^{p-2} \nabla v_2 \cdot \nabla w dx = \int_\Omega (-\Delta_p v_2) w dx = \int_\Omega -J w dx \\
\leq \int_\Omega \lambda [-f(x, J \frac{1}{p-1} \| w_\alpha \|_\infty)] w dx \\
= \int_\Omega \lambda f(x, -J \frac{1}{p-1} \| w_\alpha \|_\infty) w dx \\
\leq \int_\Omega \lambda f(x, -J \frac{1}{p-1} w_\alpha) w dx = \int_\Omega \lambda f(x, v_2) w dx;
\]

where \(x \in \Omega\) and \(w_\alpha\) is as defined before. Here in the last inequality, we use the fact that \(\frac{\partial f}{\partial u}(x, u) \geq 0\). Thus \(u_1\) is a supersolution, while \(v_2\) is a subsolution such that \(v_2 \leq v_1 \leq u_1\) and \(v_2 \leq u_2 \leq u_1\), where \(v_1\) is a strict subsolution, \(u_2\) is a strict supersolution with \(v_2 \leq 0, v_1 > 0, u_2 < 0, u_1 \geq 0\). Finally, by Lemma 2.3, (1) has at least three solutions for any \(\lambda \in I\), which completes the proof. \(\square\)

The following existence theorem is a consequence of Theorem 2.4.

**Theorem 2.5.** Let \(I = [a, b] \subset (\lambda_1, \lambda_1 + \delta), (7)\) hold, and there exist \(r > 0\) for which \(f(x, r) \leq 0, while \(f(x, -r) \geq 0\), for every \(x \in \Omega\). Assume \(r \geq (bm)^{\frac{1}{p-1}} \mu\). Then there exists at least three solutions to (1) for \(\lambda \in I\), where one of them is a positive solution, while another is a negative solution.

**Proof.** Let \(u_1(x) = (bm)^{\frac{1}{p-1}} z_\lambda(x)\) and \(u_2(x) = -v_1(x)\) be the strict sub and strict supersolutions to (1) as in the proof of Theorem 2.4. Then since \(r \geq (bm)^{\frac{1}{p-1}} \mu\), both \(v_1\) and \(u_2\) satisfy \(-r \leq u_1(x) \leq r, -r \leq u_2(x) \leq r\) (since \(\mu = \max\{|\mu_1|, |\mu_2|\}\)). But by our assumption, it is easy to see that, \(u_1(x) = r\) and \(v_2(x) = -r\) are super and subsolutions respectively to (1), because

\[-\Delta u_1(x) = -\Delta r = 0 \geq \lambda f(x, r),\]

and

\[-\Delta v_2(x) = \Delta r = 0 \leq \lambda f(x, -r).\]

Clearly, here we do not need that \(\frac{\partial f}{\partial u}(x, u) \geq 0\), and the proof is complete. \(\square\)
Remark 2.6. In [7], the authors study the existence of radial solutions to equation of the form $-\Delta_p u = \lambda f(\|x\|, u); x \in \Omega$, together with Dirichlet boundary condition, where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N (N \geq 1)$, $f$ is a sublinear function. They prove the existence result via fixed point theory. It is easy to see that if $\Omega = B_N$, where $B_N$ is the unit ball in $\mathbb{R}^N$, then under the corresponding hypothesis on $f(\|x\|, u)$ instead of $f(x, u)$, one can generate all the solutions obtained in our theorems to be radial. This follows from the fact that all the sub and supersolutions we will use in the proofs of these theorems will turn out to be radial.

References