Synchronization of Two Discrete Ginzburg-Landau Equations Using Local Coupling

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Abstract: The identical synchronization of two discrete Ginzburg-Landau equations using local coupling is proved in the theory. It is based on the theory of infinite dimensional dynamical system. It is proved that the two discrete Ginzburg-Landau equations have absorbing sets and attractors. The analytic expression of Lyapunov exponent is obtained and the results are consistent with the V. Parlitz’s numerical results. Discrete partial differential equation under local coupling is a new type of complex network.

Keywords: synchronization; Ginzburg-Landau equation; local coupling; Lyapunov exponent

1 Introduction

Synchronization phenomenon is fundamentally important in telecommunication, electronic circuits, nonlinear optics, and chemical and biological systems. In particular, synchronization of chaotic dynamics (see [1]) has attracted much attention during the last few years because of its important role in understanding the basic features of coupled nonlinear systems and its potential applications in communication systems, time series analysis and modeling (see [2]). Different coupling schemes have been proposed in order to achieve synchronization in particular for unidirectionally coupled systems (see [1]). Recently, synchronization and control of spatially extended systems such as coupled map lattices (see [3]), arrays of coupled oscillators (see [4]) or partial differential equations (PDE) (see [5]) have gained much interest. Most of the studies focus on coupled map lattices, which is the simple model for spatiotemporal chaos, but the study for synchronization of PDE is more difficult. In [5], the synchronized chaos in Geophysical Fluid Dynamics system has been investigated, the coupling method is to couple the small-scale (high-frequency) modes, while leaving the large-scale modes uncoupled. The synchronization behavior also governs the relationship between different sectors of the same continuous channel. As to the related phenomenon of controlled chaos, which is recently shown to apply to a model of the El Niño cycle, synchronized chaos is shown to be relevant to the behavior of the Earth’s atmosphere. But this coupling is more difficult to realize in physics. In [6], the coupling scheme is used for PDE’s based on local spatial averaged coupling signals (sensors), which is a model for typical experimental sensors. The theoretical interpretation can not be given only using numerical examination.

In this paper we prove the synchronization for coupled Ginzburg-Landau equations using local coupling. Consider the following drive system, one dimensional complex Ginzberg-Landau equation (GLE)

\[
\frac{\partial u}{\partial t} = \mu u - (k + i\beta)|u|^2 u + (\lambda + i\alpha)\Delta u ,
\]

(1.1)
Let \( x \in \Omega = [0, L] \), with periodic boundary condition \( u(x + L, t) = u(x, t) \).

The response system is

\[
\frac{\partial v}{\partial t} = \mu v - (k + i\beta)|v|^2v + (\lambda + i\alpha)\Delta v + f(\tilde{u}_n, \tilde{v}_n),
\]

with periodic boundary condition \( v(x + L, t) = v(x, t) \).

where

\[
\tilde{u}_n(t) = \frac{1}{t} \int_{nd-rac{l}{2}}^{nd+rac{l}{2}} u(x, t)dx, \quad n = 1, 2, \cdots, N,
\]

which represents average of spatial intervals with width \( l \). [see Fig1(a),(b)]. We use a unidirectional dissipative coupling. To implement this coupling we want to measure \( N \) sensor signals of the drive system at the same positions and apply the dissipative coupling term with coupling strength \( \varepsilon \)

\[
f(\tilde{u}_n, \tilde{v}_n) = \begin{cases} 
\varepsilon(\tilde{u}_n - \tilde{v}_n), & \text{nd} - \frac{1}{2} \leq x \leq \text{nd} + \frac{1}{2}, \\
0, & \text{elsewhere},
\end{cases}
\]

at each sensor position \( n = 1, 2, \cdots, N \), where \( d = \frac{L}{N} \).

Let \( m \in N \), \( h = \frac{L}{m} \), \( u_n = u(nh, t) \), \( v_n = v(nh, t) \), substituting to (1.1),(1.2), we have

\[
\begin{align*}
\dot{u}_n &= \mu u_n - (k + i\beta)|u_n|^2u_n + (\lambda + i\alpha)\frac{1}{h^2}(u_{n+1} - 2u_n + u_{n-1}), \\
\dot{v}_n &= \mu v_n - (k + i\beta)|v_n|^2v_n + (\lambda + i\alpha)\frac{1}{h^2}(v_{n+1} - 2v_n + v_{n-1}) + f(\tilde{u}_n, \tilde{v}_n),
\end{align*}
\]

\((n = 1, 2, \cdots, n)\) with periodic boundary conditions \( u_{n+m} = u_n, v_{n+m} = v_n \), where

\[
\tilde{u}_n = \frac{1}{l} \sum_{n-k\frac{l}{2}}^{n+k\frac{l}{2}} u_n(t), \quad f_k(\tilde{u}_n, \tilde{v}_n) = \begin{cases} 
\varepsilon(\tilde{u}_n - \tilde{v}_n), & \text{nd} - \frac{1}{2} \leq x \leq \text{nd} + \frac{1}{2}, \\
0, & \text{elsewhere},
\end{cases}
\]

\(l < n_1\) is even number, \( k = 1, 2, \cdots, n \), the distance of two sensors is larger than \( 4h \).

Let \( M \) be \( m \) dimensional vector space with periodic boundary condition, we give the following definitions:

**Definition1.** Let \( v = (v_1, v_2, \cdots, v_m) \in M \), define difference by \((Dv)_i = \frac{v_i - v_{i-1}}{h}\), and operator \( D \) by

\[
(Dv) = \frac{1}{h} \begin{pmatrix}
1 & 0 & \cdots & \cdots & -1 \\
-1 & 1 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
\vdots & \vdots & \cdots & -1 & 0
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_m
\end{pmatrix}.
\]

**Definition2.** Define \((Av)_i = \frac{1}{h^2}(2v_i - v_{i+1} - v_{i-1})\), and operator \( A \) by

\[
(Av) = \frac{1}{h^2} \begin{pmatrix}
2 & -1 & \cdots & \cdots & -1 \\
-1 & 2 & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
\vdots & \vdots & \cdots & -1 & 0
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_m
\end{pmatrix}.
\]

**Definition3.** Define the norms by

\[
\|v\| = \left( h \sum_{i=1}^{m} |v_i|^2 \right)^{\frac{1}{2}}, \quad \|v\| = \|Dv\| = \left( h \sum_{i=1}^{m} \left( \frac{v_i - v_{i-1}}{h} \right)^2 \right)^{\frac{1}{2}},
\]

\[
\|A\| = \left( h \sum_{i=1}^{m} \left( \frac{2v_i - v_{i+1} - v_{i-1}}{h} \right)^2 \right)^{\frac{1}{2}},
\]

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Figure 1: Left: Principle of the sensor coupling scheme. Right: Visualization of three sensor time series (plotted overlayed) measured from spatiotemporal chaos.

Figure 2: Dynamical regimes of the Ginzburg-Landau equation (4). (a) Phase turbulence. (b) Defect turbulence, amplitudes are gray scaled.

\[ \langle v, \omega \rangle = \frac{1}{h} \sum_{i=1}^{m} v_i \omega, \quad \langle Dv, D\omega \rangle = \frac{1}{h} \sum_{i=1}^{m} (v_i - v_{i-1})(\omega_i - \omega_{i-1}). \]

It is easily proved that
\[ \langle Av, \omega \rangle = \langle Dv, D\omega \rangle. \quad \text{(1.11)} \]

**Definition 4.** Two discrete Ginzburg-Landau Equations under local coupling are called identical synchronization, if their states converge to each other in the whole spatial domain, i.e.

\[ \lim_{t \to \infty} \| u(x, t) - v(x, t) \| = \lim_{t \to \infty} \left( \frac{1}{h} \sum_{n=0}^{m} |u_n(t) - v_n(t)|^2 \right)^{\frac{1}{2}} = 0. \]

In [6], taking \( k = 1, \alpha = \tilde{\beta}, \lambda = 1, \beta = \alpha \), this equation possesses uniform traveling wave solutions, which are unstable for parameter values of \( \alpha \) and \( \beta \) with \( 1 - \alpha \beta < 0 \) where different types of turbulence occur, \( \mu = 1.0, \alpha = 2.0, \beta = 0.7 \), corresponding to phase turbulence and \( \mu = 1.0, \alpha = 2.0, \beta = 1.2 \), corresponding to defect turbulence.\[\text{see Fig2(a),(b)}\].

In Fig3. [6] use \( N = 15 \) equally spaced sensors with width \( l = 3 \) and coupling strength \( \varepsilon = 0.2 \) to synchronize two Ginzburg-Landau Equations with length \( L = 100 \) in the phase turbulent regime. The relation between these coupling parameters \( N, l, \varepsilon \) is obtained by using numerical computation.\[\text{see Fig4,5,6}\].

In this paper we prove the previous numerical results for synchronization by using infinite dimensional dynamical system theory. In sec. 2, the existence of absorbing sets and attractors of (1.5),(1.6) is proved. In sec. 3, the analytic expression of normal Lyapunov exponent is obtained. In sec. 4, from estimate we can obtain the relation of the three coupling parameters \( N, l, \varepsilon \), which is in agreement with Fig4,5,6.

## 2 Absorbing Set and Attractor

We first study the properties of operator \( A : M \to M \)

**Lemma 2.1.** \( A \) is a symmetric matrix, so it can be diagonalized. 0 is an eigenvalue of \( A \) with the corre-
Figure 3: Synchronization of two coupled GLE’s in the phase turbulent regime. (a) Drive system. (b) Response system driven by $N = 15$ sensors with $l = 3$ and coupling strength $\varepsilon = 0.2$.

Figure 4: Lyapunov dimension $D_L$ vs the system length $L$ for defect turbulence [(a) $\mu = 1.0, \alpha = 2.0, \beta = 1.2$] and phase turbulence [(b) $\mu = 1.0, \alpha = 2.0, \beta = 0.7$].

Figure 5: Minimal number $N$ of coupling signals needed for synchronization vs the spatial length $L$ for fixed width $l$ of the sensors. Left: $l = 0.5$ and $l = 3.0$. The coupling strength $\varepsilon$ increases from top to bottom $\varepsilon = 0.5, 1.0, 2.0, 3.0, 4.0$.

Figure 6: The left figure shows the minimal number $N$ of sensors needed for synchronization vs the width $l$ of the sensors for a fixed system length $L = 100$. The right plot shows the number $N$ of sensors needed for synchronization normalized to the Lyapunov dimension $D_L$ vs the width $l$. The coupling strength $\varepsilon$ increases from top to bottom $\varepsilon = 0.5, 1.0, 1.5, 2.0, 3.0, 4.0$. The dotted line in the left plot indicates the border where the sensors begin to overlap.

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Lemma 2.2. For any $0 \leq k \leq m-1$, $1 \leq i \leq m$ define

$$e(k, i) = \begin{cases} 1, & k = 0, \\ \sin \left( \frac{2k\pi i}{m} \right), & 1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor, \\ \cos \left( \frac{2(m-k)\pi i}{m} \right), & \left\lfloor \frac{m}{2} \right\rfloor \leq k \leq m-1. \\ \end{cases}$$

(2.13)

where $\left\lfloor s \right\rfloor$ is the largest integer not greater than $s$, $e(k) = (e(k, 1), e(k, 2), \ldots, e(k, m))$. Then for $0 \leq k \leq m-1$ we have

$$Ae(k) = \frac{4}{h^2} \sin^2 \left( \frac{k\pi}{m} \right) e(k).$$

(2.14)

Proof: Suppose $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor$,

$$h^2 A(e(k)) = \begin{pmatrix} 2 & -1 & \cdots & -1 \\ -1 & 2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & \cdots & -1 & 2 \end{pmatrix} \begin{pmatrix} e(k, 1) \\ e(k, 2) \\ \vdots \\ e(k, m) \end{pmatrix},$$

and the corresponding eigenvalues are

$$\lambda(k) = \frac{4}{h^2} \sin^2 \left( \frac{k\pi}{m} \right).$$

(2.15)

Corollary 2.2. Denote by $e = (1, 1, \cdots, 1)$, $e^\perp = \{ v \in M \mid \langle e, v \rangle = 0 \}$. Then $Au = 0, \forall u \in \text{Span} \{ e \}$. And if $m \geq 2, \forall v \in e^\perp$, $(Av, v) = \| Dv \|^2 \geq \frac{4}{h^2} \sin^2 \left( \frac{\pi}{m} \right) \| v \|^2 \geq \frac{16}{h^2} \| v \|^2$.

Proof: Since $\sin \frac{\pi x}{m}$ is decreasing in $\left(0, \frac{\pi}{m}\right)$.

Definition 5. For $v \in M$, $\forall p, q \in [1, \infty)$,

$$|v|_q = \left( \sum_{n=1}^{m} |v_n|^q \right)^{\frac{1}{q}} , \quad |v|_{1,p} = \left( \sum_{n=1}^{m} (|v_n|^p + |Dv_n|^p) \right)^{\frac{1}{p}} .$$

Lemma 2.3. (Discretized Sobolev Inequality). For any $q \in [1, \infty)$, the following embedding $(M, | \cdot |_{1,2}) \rightarrow (M, | \cdot |_q)$ is true. In other words, there exists a constant $k_2$, independent of $m, L$ such that

$$|v|_q \leq K_2 |v|_{1,2} .$$

(2.16)

Lemma 2.4. There exists a constant $k_3$, such that

$$|u|_4 \leq K_3 \| u \|^{\frac{1}{2}} \| u \|^{\frac{1}{1,2}} , \quad (\forall u \in M).$$

(2.17)

Lemma 2.5. (Discretized Lieb-Thirring inequality) Let $\varphi^{(l)}$, $1 \leq l \leq \tilde{N} \leq m$ be a family in $M$ which is orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle$.

Let $\rho = (\rho_1, \rho_2, \cdots, \rho_m) \in M$, $\rho_n = \sum_{l=1}^{\tilde{N}} |\varphi_n^{(l)}|^2$, $n = 1, 2, \cdots, m$, for every $p$, $1 < p \leq 2$, there exists a constant $k_4 > 0$, independent of $m$ such that

$$\left( \sum_{n=1}^{m} \rho_n^{\frac{p}{p-1}} \right)^{p-1} \leq K_4 \left( \sum_{l=1}^{\tilde{N}} \| \varphi^{(l)} \|_{1,2}^2 + 1 \right) .$$

(2.18)

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The proof of these lemmas see [7].

We first prove the existence of an absorbing set in $M$. The necessary energy equation is obtained by multiplying (1.5) by $\pi = (\pi_1, \pi_2, \cdots, \pi_m)$, the conjugate of $u$, adding from 1 to $m$, and taking the real part of the equation, we obtain

$$\frac{1}{2} \frac{d}{dt} \| u \|^2 + \lambda \| u \|^2 + k |u|_4^4 - \mu \| u \|^2 = 0,$$

with inequality

$$ks^4 - 4\mu s^2 \geq -\frac{4}{k} \mu^2, \forall s \in R,$$

yields

$$\frac{d}{dt} \| u \|^2 + 2\lambda \| u \|^2 + k |u|_4^4 + 2\mu \| u \|^2 \leq \frac{4\mu^2}{k} L,$$  \hspace{1cm} (2.19)

then with the Gronwall lemma

$$\| u(t) \|^2 \leq \| u(t_0) \|^2 \exp(-2\mu t) + \frac{1}{2\mu} \left[ \frac{4\mu^2 L}{k} \right] (1 - \exp(-2\mu t)), $$

$$\lim_{t \to \infty} \sup \| u(t) \|^2 \leq \rho_{10}^2, \rho_{10}^2 = \left[ \frac{2\mu L}{k} \right].$$  \hspace{1cm} (2.20)

Therefore, the ball of $M$, $B_0 = B_M(0, \rho_{10})$ (centered at 0 with radius $\rho_{10} > \rho_{01}$) is positively invariant for the semigroup $S_1(t) : u_0 \to u(t)$, $B_0$ is absorbing in $M$ for the semigroup $S_1(t)$. If $B$ is a bounded set of $M$, in other words, $B$ is the ball of $M$ centered at 0 with radius $R$ denoted by $B_M(0, R)$, then for $t \geq t_0 = t_0(B, B_0)$, $t_0 = \frac{1}{2\mu} \log \frac{R^2}{\rho_{01}^2 - \rho_{10}^2}$, $S_1(t)B \subset B_0$.

Integrate (1.5) between $t$ and $t + \bar{\tau} (\bar{\tau} > 0)$, if $u_0 \in B$, $t \geq t_0$, we obtain

$$\int_{t}^{t + \bar{\tau}} \left\{ 2\lambda \| u \|^2 + k |u|_4^4 + 2\mu \| u \|^2 \right\} ds \leq \rho_{10}^2 + \frac{4\mu^2 \bar{\tau}}{k} L.$$  \hspace{1cm} (2.21)

Analogous to (1.5) , from (1.6) , we obtain

$$\frac{1}{2} \frac{d}{dt} \| v \|^2 + \lambda \| v \|^2 + k |v|_4^4 - \mu \| v \|^2 = Re \sum_{k=1}^{m} f(\bar{u}_k, \bar{v}_k) \pi_k \leq \varepsilon N \delta_1 \| u \|^2 + \varepsilon N \delta_2 \| v \|^2,$$

(where $\delta_1, \delta_2$ depend on $N, l$).

$$\frac{d}{dt} \| v \|^2 + 2\lambda \| v \|^2 + k |v|_4^4 + 2(\mu + \varepsilon \delta_2 N) \| v \|^2 \leq \frac{4(\mu + \varepsilon \delta_2 N)^2}{k} L + \varepsilon N \delta_1 \| u \|^2,$$  \hspace{1cm} (2.22)

$$\| v(t) \|^2 \leq \| v_0 \|^2 \exp(-2\mu - 2\varepsilon \delta_2 N)t + \frac{\left( \frac{1}{k} \right)(\mu + \varepsilon \delta_2 N)^2 L + (\varepsilon N \delta_1 \rho_{10}^2)}{2(\mu + \varepsilon \delta_2 N)} (1 - \exp(-2\mu - 2\varepsilon \delta_2 N)t),$$

$$\lim_{t \to \infty} \sup \| v(t) \|^2 \leq \rho_{20}^2, \rho_{20}^2 = \frac{\left( \frac{1}{k} \right)(\mu + \varepsilon \delta_2 N)^2 L + (\varepsilon N \delta_1 \rho_{10}^2)}{2(\mu + \varepsilon \delta_2 N)}.$$  \hspace{1cm} (2.23)

Therefore, the ball of $M$, $B_0 = B_M(0, \rho_{20})$ (centered at 0 with radius $\rho_{20} > \rho_{20}$) is positively invariant for $S_2(t) : v_0 \to v(t)$, $B_0'$ is absorbing in $M$ for the semigroup $S_2(t)$, then

$$t \geq t_0 = t_0(B, B_0), t_0 = \frac{1}{2R} \log \frac{R^2}{(\rho_{20}^2 - \rho_{20}), \hspace{1cm} S_2(t)B \subset B_0'}$$

Integrate (2.11) between $t$ and $t + \bar{\tau} (\bar{\tau} > 0)$, if $v_0 \in B$, $t \geq t_0$ we obtain

$$\int_{t}^{t + \bar{\tau}} \left\{ 2\lambda \| v \|^2 + k |v|_4^4 + 2(\mu + \varepsilon \delta_2 N) \| v \|^2 \right\} ds \leq \rho_{20}^2 + \frac{4(\mu + \varepsilon \delta_2 N)^2 \bar{\tau}}{k} L + \varepsilon N \delta_1 \rho_{10}^2 \bar{\tau}.$$
Then, we show the existence of an absorbing set for \( \| \cdot \|_1 \). We multiply (1.5) by \( A\overline{\pi} \), adding from 1 to \( m \), obtain,

\[
\frac{1}{2} \frac{d}{dt} \| u \|_1^2 + \lambda |Au|^2 - \mu \| u \|_1^2 = Re (k + \bar{\beta}i)h \sum_{n=1}^{m} |u_n|^2 u_n A\overline{\pi} \leq 3 \left[ (k)^2 + \bar{\beta}^2 \right]^\frac{1}{2} |u|_1^2 |Du|_1^2. \tag{2.24}
\]

By the discretized Sobolev embedding and the interpolation inequalities

\[
|\phi|_4 \leq K_4 \| \phi \|_2^\frac{3}{4} \| \phi \|_1^\frac{1}{4},
\]

\[
|D\phi|_4 \leq K_4 \| D\phi \|_2^\frac{3}{4} \| A\phi \|_1^\frac{1}{4}.
\]

These allow us to majorize (2.13) by

\[
\frac{\lambda}{2} \| Au \|^2 + \frac{9}{4\lambda} K_4^2 (k^2 + \bar{\beta}^2) \| u \|_4^4 \| u \|_1^2.
\]

Let \( C'_3 = \frac{9}{2\lambda} (C_2')^4 \left[ k^2 + \bar{\beta}^2 \right] \), we have

\[
\frac{d}{dt} \| u \|_1^2 \leq 2(\mu + C'_3 |u|_4^4) \| u \|_1^2.
\]

Now, we apply the uniform Gronwall lemma to

\[
y = \| u \|^2, \quad g = 2(\mu + C'_3 |u|_4), \quad h(s) = 0,
\]

\[
\int_t^{t+\bar{\tau}} g(s)ds \leq a_1, \quad \int_t^{t+\bar{\tau}} h(s)ds \leq a_2, \quad \int_t^{t+\bar{\tau}} y(s)ds \leq a_3,
\]

we obtain

\[
\| u(t) \|_1^2 \leq \left( \frac{a_3}{\bar{\tau}} + a_2 \right) \exp a_1 \quad (t \geq t_0 + \bar{\tau}),
\]

where \( \bar{\tau} > 0 \) is arbitrary, \( u_0 \in B \). If \( B \) is a bounded set for \( \| \cdot \|_1 \) in \( M \), then it is also a bounded set of \( M, S_1(B) \subset 0, (t \geq t_0(B, B_0); S_1(t)B \subset B_1, (t \geq t_0 + \bar{\tau}) \), where \( B_1 \) is the ball of \( M \) centered at 0 with radius \( \rho_{11} \)

\[
\rho_{11}^2 = \rho_{10}^2 + \left( \frac{a_3}{\bar{\tau}} + a_2 \right) \exp a_1.
\]

\( B_1 \) is an absorbing set of the semigroup \( S_1(t) \) in \( M \) for \( \| \cdot \|_1 \).

Analagous to (5), multiply (6) by \( A\overline{\pi} \), adding from 1 to \( m \),

\[
\frac{1}{2} \frac{d}{dt} \| v \|^2 + \lambda |Av|^2 - \mu \| v \|^2 = Re (k + \bar{\beta}i)h \sum_{n=1}^{m} |v_n|^2 v_n A\overline{\pi} + Re h \sum_{n=1}^{m} f(u_n, v_n) A\overline{\pi} \leq 3 \left[ (k)^2 + \bar{\beta}^2 \right]^\frac{1}{2} |v|_1^2 |Dv|_1^2.
\]

The same conclusion is proved.

**Theorem 2.6.** We consider the dynamical system associated with the two discrete Ginzburg-Laudau equations (1.5)-(1.6) with periodic boundary conditions, then this dynamical system possesses an attractor \( \Theta \) which is compactly connected and maximal in \( M \times M \).

### 3 Estimate of Lyapunov Exponent

Consider as flow \( \phi_t(z), z = (u, v) \), where \( u, v \) represent state variables of two coupled systems. We say that two coupled systems are in the state of generalized chaos synchronization if there exists a chaotic attractor \( \Theta \) and a function \( \Phi : M \times M \to M \times M \) such that \( \Phi = \{(u, v) : \Phi(u, v) = 0\} \) is a stable (i.e. attracting) invariant manifold, \( \Theta \) is a subset of \( \Phi \). In particular, if \( \Phi = \{(u, v) : u = v\} \), then identical synchronization occurs between the two subsystems. There are two linear mutually orthogonal spaces at
each point \( z \in \tilde{M} \), the tangent space \( T_z\tilde{M} \) and the normal space \( N_z\tilde{M} \). Let \( P : T_z\tilde{M} \times N_z\tilde{M} \to N_z\tilde{M} \) be the orthogonal projection to the normal subspace \( N_z\tilde{M} \). Now we consider the linear part, \( D\phi_t(z) \) of \( \phi_t(z) \) in the invariant manifold \( M \).

Let \( v(0) \in T_z\tilde{M}, \, v(t) = D\phi_t(z)v(0); \, w(0) \in N_z\tilde{M}, \, w(t) = P D\phi_t(z)w(0). \)

The invariant manifolds \( \tilde{M} \) is stable, if \( \lim_{t\to\infty} \|w(t)\| = 0 \), for all \( z \in \tilde{M} \) and all vectors \( w(0) \in N_z(\tilde{M}) \). It is said to be normally hyperbolic if \( \lim_{t\to\infty} \|w(t)\| = 0 \), for all \( z \in \tilde{M} \) and all nonzero vectors \( \omega \in N_z(\tilde{M}) \), \( v \in T_z(\tilde{M}) \). Normal hyperbolicity is a necessary and sufficient condition for the persistence of the invariant manifold under small arbitrary perturbations of the system.(see\[8\]).

The condition for stability and normal hyperbolicity can be expressed in terms of Lyapunov exponents. Let \( \rho \) be an ergodic invariant measure supported in \( \tilde{M} \), then there exist \( m \) tangential Lyapunov exponents(LE’s), \( n - m \) normal LE’s. We let \( \lambda_{\text{max}}(\rho) \) as the largest normal LE and \( \mu_{\text{min}}(\rho) \) as the smallest tangential LE , and define

\[
\lambda_{\text{max}} = \sup_{\rho \in E} \lambda_{\text{max}}(\rho), \quad \lambda_{\text{min}} = \inf_{\rho \in E} \lambda_{\text{max}}(\rho),
\]

\[
\mu_{\text{max}} = \sup_{\rho \in E} \mu_{\text{min}}(\rho), \quad \mu_{\text{min}} = \inf_{\rho \in E} \mu_{\text{min}}(\rho),
\]

where \( E \) is an ergodic invariant probability measure supported in \( \tilde{M} \).

The invariant manifold is stable iff \( \lambda_{\text{max}} < 0 \). And the stable manifold is normally hyperbolic if \( \lambda_{\text{max}} < \mu_{\text{min}} \).

On the basis of the criterions in \[8\], we study the parameter conditions for the identical synchronization of (1.5),(1.6). Let \( w = u - v \), then \( w \) satisfies the first variation equation

\[
\frac{d\tilde{w}_n}{dt} = \frac{dF(u)}{dt}w_n, \tag{3.26}
\]

\[
(F'(u)w)_n = -(\lambda + i\alpha)Aw_n + (k + i\tilde{\beta}) \left\{ |u_n|^2w_n + 2u_nRe(\tilde{w}_n\tilde{w}_n) \right\} + \mu w_n - \varepsilon Tw_n,
\]

with periodic boundary condition and initial value \( w(0) = \xi \). It is easily to show that for every \( \xi \in M \), (3.2) possesses a unique solution \( w \).

For \( \tilde{m} \in N \), we consider \( \xi = \xi_1, \xi_2, \ldots, \xi_{\tilde{m}} \) (the elements of \( M \)), and the corresponding solutions \( w = w_1, w_2, \ldots, w_{\tilde{m}} \) of (3.2), \( w = u(\tau) = S_1(\tau)u_0 \) is a fixed orbit on the synchronized manifold. We have

\[
|w_1(t) \wedge \cdots \wedge w_{\tilde{m}}(t)|_{\lambda, M} = |\xi_1 \wedge \cdots \wedge \xi_{\tilde{m}}| \exp \int_0^t Re Tr F'(S_1(\tau)u_0) \circ Q_{\tilde{m}}(\tau) d\tau,
\]

where \( Q_{\tilde{m}}(\tau) = Q_{\tilde{m}}(\tau, u_0, \xi_1, \ldots, \xi_{\tilde{m}}) \) is a orthogonal projector from \( M \) to the space spanned by \( w_1(\tau), \ldots, w_{\tilde{m}}(\tau) \). At a given time \( \tau \), let \( \varphi_j(\tau), j \in N \), be an orthonormal basis of \( M \) such that \( \varphi_1(\tau), \ldots, \varphi_{\tilde{m}}(\tau) \) spanned \( Q_{\tilde{m}}(\tau)M = \text{Span}[w_1(\tau), \ldots, w_{\tilde{m}}(\tau)] \):

\[
Re Tr F'(u(\tau)) \circ Q_{\tilde{m}}(\tau) = \sum_{j=1}^{\tilde{m}} Re \left( F'(u(\tau)) \varphi_j(\tau), \varphi_j(\tau) \right)
\]

Omitting temporary variable \( \tau \), we see that

\[
Re \left( F'(u(\tau)) \varphi_j, \varphi_j \right) = -\lambda \left\| \varphi_j \right\|^2_1 + kh \sum_{n=1}^{\tilde{m}} |u_n|^2 \left\| \varphi_n \right\|^2_k + 2h \sum_{n=1}^{\tilde{m}} Re \overline{u_n} \varphi_n^{(j)} \left\{ \tilde{\beta} I_m \left( u_n \overline{\varphi_n^{(j)}} \right) - k Re \left( u_n \varphi_n^{(j)} \right) \right\} + \mu \left\| \varphi_j \right\|^2 + \varepsilon \frac{1}{1} \sum_{k=1}^{N} n_k + \frac{1}{1} \sum_{k=1}^{N} \left\| \varphi_n \right\|^2_k.
\]
Since \(\{\varphi^{(j)}\}\) being orthonormal, \(\|\varphi^{(j)}\| = 1\) and
\[
 h \sum_{n=1}^{m} \tilde{\beta} Re \left( \overline{\varphi_n^{(j)}} \varphi_n^{(j)} \right) \leq \tilde{\beta} \left( h \sum_{n=1}^{m} |u_n|^2 \right) \left( h \sum_{n=1}^{m} |\varphi_n^{(j)}|^2 \right),
\]
hence
\[
 Re \left( F'(u)\varphi^{(j)}, \varphi^{(j)} \right) \leq -\lambda \left( h \sum_{n=1}^{m} |u_n|^2 \right) \left( h \sum_{n=1}^{m} |\varphi_n^{(j)}|^2 \right) + \mu \varepsilon \frac{1}{N} \sum_{k=1}^{N} \left( \sum_{n_k-\frac{1}{2}}^{n_k+\frac{1}{2}} |\varphi_n^{(j)}|^2 \right),
\]
\[
 Re \sum_{j=1}^{\hat{m}} \left( F'(u)\varphi^{(j)}, \varphi^{(j)} \right) \leq -\lambda \sum_{j=1}^{\hat{m}} \left( h \sum_{n=1}^{m} |u_n|^2 \right) \left( h \sum_{n=1}^{m} |\varphi_n^{(j)}|^2 \right) + \mu \hat{m} \varepsilon \frac{1}{N} \sum_{k=1}^{N} \left( \sum_{n_k-\frac{1}{2}}^{n_k+\frac{1}{2}} |\varphi_n^{(j)}|^2 \right). \tag{3.27}
\]
Let \(\rho_n = \sum_{j=1}^{\hat{m}} |\varphi_n^{(j)}|^2\), by the discrete Lieb-Thirring inequality
\[
 \left( h \sum_{n=1}^{m} \rho_n^{\frac{p}{2}+1} \right)^{p-1} \leq K_5 \left( \sum_{n=1}^{m} |\varphi_n^{(j)}|^2 + 1 \right),
\]
taking \(p = \frac{3}{2}\), from Holder and Young inequality
\[
 2 \tilde{\beta} \left( h \sum_{n=1}^{m} |u_n|^2 \rho \right) \leq 2\tilde{\beta} |u_n|^3 |\rho|_3,
\]
\[
 h \sum_{n=1}^{m} |\rho_n| = h \sum_{n=1}^{m} \sum_{j=1}^{\hat{m}} |\varphi_n^{(j)}| = \hat{m},
\]
\[
 \hat{m} = h \sum_{n=1}^{m} |\rho_n| \leq h \left( \frac{m}{n=1} |\rho_n|^3 \right)^{\frac{1}{3}} \left( \sum_{n=1}^{m} \frac{1}{3} \right)^{\frac{2}{3}} = h \left( \sum_{n=1}^{m} |\rho_n|^3 \right)^{\frac{1}{3}} \left( hm \right)^{\frac{2}{3}},
\]
and we can majorize the right side of (3.3)
\[
 -\lambda \left\{ \frac{1}{K_5} \left( h \sum_{n=1}^{m} \rho_n^{\frac{3}{2}} \right)^{\frac{1}{2}} - 1 \right\} + 2 \tilde{\beta} |u_n|^3 |\rho|_3 + \mu \hat{m} \varepsilon \frac{1}{N} \sum_{k=1}^{N} \left( \sum_{n_k-\frac{1}{2}}^{n_k+\frac{1}{2}} |\varphi_n^{(j)}|^2 \right) \leq -\lambda \frac{\tilde{\beta}^3}{2K_5} |\rho|_3^3 + \lambda + \frac{8 |\tilde{\beta}|^3 |u_n|^6}{3 \left( \frac{\lambda^2}{|u_n|^2} \right)} + \mu \hat{m} \varepsilon \frac{1}{N} \sum_{k=1}^{N} \left( \sum_{n_k-\frac{1}{2}}^{n_k+\frac{1}{2}} |\varphi_n^{(j)}|^2 \right) \leq -\lambda \frac{\tilde{\beta}^3}{2K_5} |\rho|_3^3 + \lambda + \frac{24k_3^2 |\tilde{\beta}|^3 |u_n|^6}{\lambda^2} + \mu \hat{m} \varepsilon \frac{1}{N} \sum_{k=1}^{N} \left( \sum_{n_k-\frac{1}{2}}^{n_k+\frac{1}{2}} |\varphi_n^{(j)}|^2 \right). \tag{3.28}
\]
Assuming now that \( u_0 \) belongs to the global attractor \( \mathcal{Y} \), we can majorize the quantity

\[
q_{\hat{m}} \leq \sum_{t \to \infty} \sup_{\xi_i \in H} \sup_{|\xi| \leq 1, i=1, \ldots, m} \frac{1}{t} \int_0^t Re(T_\tau F'(S(\tau)u_0) \circ Q_m(\tau)) d\tau
\]

\[
\leq -k_1 \hat{m}^2 + k_2 + \mu \hat{m} - \varepsilon W_{\hat{m}} + \lambda ,
\]

where

\[
\lambda = \frac{\lambda}{2 K_3 L}, \quad k_2 = \frac{24 k_3^2}{\lambda^2} \delta ,
\]

\[
\delta = \limsup_{t \to \infty} \sup_{u_0 \in \mathcal{Y}} \frac{1}{t} \int_0^t \| u(\tau) \|_N^2 d\tau ,
\]

\[
W_{\hat{m}} = \liminf_{t \to \infty} \frac{1}{t} \int_0^t \sum_{j=1}^{\hat{m}} \frac{1}{l} \sum_{j=1}^N \left| \sum_{k=-j}^{j} \phi_{ij}^{(j)} \right|^2 d\tau .
\]

We infer (3.5) the following bound on the uniform normal Lyapunov exponents \( \mu_j, j \in N \), with synchronized manifold

\[
\mu_1 + \mu_2 + \cdots + \mu_j \leq q_j \leq -k_1 j^{\frac{3}{2}} + k_2 + \mu j - \varepsilon W_j + \lambda .
\]

We take these parameters , which satisfy

\[
\mu_1 \leq q_1 \leq -k_1 + k_2 + \mu - \varepsilon W_1 + \lambda ,
\]

\[
\mu_1 + \mu_2 \leq q_2 \leq -k_1 2^{\frac{3}{2}} + k_2 + 2\mu - \varepsilon W_2 + \lambda ,
\]

\[
\cdots 
\]

\[
\mu_1 + \mu_2 + \cdots + \mu_j \leq q_j \leq -k_1 j^{\frac{3}{2}} + k_2 + \mu j - \varepsilon W_j + \lambda .
\]

All normal Lyapunov exponents could be negative numbers, and the invariant manifold is stable , (1.1),(1.2)realize identical synchronization.

When \( \varepsilon = 0 \), we obtain that tangential Lyapunov exponents satisfy the following estimate expression:

\[
\tilde{\mu}_1 + \tilde{\mu}_2 + \cdots + \tilde{\mu}_j \leq q_j \leq -k_1 j^{\frac{3}{2}} + k_2 + \mu j + \lambda ,
\]

if \( \hat{m} \) is defined by

\[
\tilde{\mu}_1 + \tilde{\mu}_2 + \cdots + \tilde{\mu}_{\hat{m}} \leq q_{\hat{m}} \leq -k_1 \hat{m}^3 + k_2 + \mu \hat{m} + \lambda < 0 ,
\]

then the Hausdorff dimension of \( \mathcal{Y} \) is less than or equal to \( \hat{m} \). Since Hausdorff dimension is closer to Lyapunov dimension , the Lyapunov dimension of \( \mathcal{Y} \) still is less than or equal of \( \hat{m} \).

Let \( \lambda_{\text{max}} = \max \{ \mu_1, \mu_2, \cdots, \mu_m \} \), \( \lambda_{\text{min}} = \min \{ \mu_1, \mu_2, \cdots, \mu_m \} \), from (3.6)and (3.7) we can adjust the parameters \( \varepsilon, L, l \) such that \( \lambda_{\text{max}} < \lambda_{\text{min}} \), the stable synchronized manifold is normally hyperbolic .

4 Discussion : the relation between the coupling parameters for synchronization of Ginzberg-Landau equations

(1) The relation of Lyapunov dimension \( D_L \) and the system length \( L \)
From (3.7), we see that the larger \( L, \) the larger \( \frac{L^2}{k_1} \), \( m \) must be increased to keep (3.7), i.e. the Hausdorff dimension increases , so the Lyapunov dimension still increases. It is identical with Fig 4 .

(2) The relation between minimal number \( N \) of coupling signals and the spatial length \( L \) for fixed width 1 of the sensors and the coupling strength \( \varepsilon : \)
From (3.6), we see that the larger \( L, \) the smaller \( k_1, k_2 \) is almost unchanged and the larger \( \frac{L^2}{k_1} \), then minimal number \( N \) of the sensors must be increased to keep (3.6). It is identical with Fig5. For a fixed \( L, N \), we see that the larger \( \varepsilon \), it may decrease number \( N \) of the sensors to keep(3.6).It is identical with Fig5.

(3) The relation of \( N \) and \( l \) for a fixed system length \( L \)
From (32),we see that the large \( N, \)the more small \( l \), it must decrease \( l \) to keep (3.6).It is identical with Fig6.

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5 Conclusion

In this paper we discuss synchronization for discrete continuous spatially extended systems using the theory of infinite dimensional dynamical system (here it is the synchronization of Ginzberg-Landau equations), and obtain the expression of normal Lyapunov exponents and the relation between system parameters. The method can be applied to another kind of PDE, which will be published in further paper.

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References