The Dullin-Gottwald-Holm Equation: Classical Lie Approach and Exact Solutions

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Abstract: The Lie-group formalism is applied to investigate the symmetries of the Dullin-Gottwald-Holm equation
\[ \phi_t - \alpha^2 \phi_{xx,t} + 2w\phi_x + 3\phi\phi_x + \gamma\phi_{xxx} = \alpha^2 (2\phi_x\phi_{xx} + \phi\phi_{xxx}) \],
which describes the unidirectional propagation of two dimensional waves in shallow water over a flat bottom. We derived the infinitesimals that admit the classical symmetry group. The reduced ordinary differential equation is further studied and many new families of traveling wave solutions are successfully obtained.

Keywords: Dullin-Gottwald-Holm equation; Lie Classical method; exact solution

1 Introduction

The nonlinear partial differential equation [1]
\[ m_t + c_0m_x + \phi m_x + 2m\phi_x = -\gamma\phi_{xxx}; x \in \mathbb{R}, t \in \mathbb{R}. \]  
(1.1)
in dimensionless time-space variables \((t, x)\) models the unidirectional propagation of two dimensional waves in shallow water over a flat bottom. In (1.1), \(\phi(t, x)\) represents the horizontal component of the fluid velocity, \(m = \phi - \alpha^2 \phi_{xx}\) is a momentum variable, the constants \(\alpha^2\) and \(\gamma/c_0\) are squares of length scales, and \(c_0 = \sqrt{gh}\) (where \(c_0 = 2w\)) is the linear wave speed for undisturbed water at rest at spatial infinity. In [1], Dullin, Gottwald and Holm derived the equation (1.1) by using asymptotic expansions directly in the Hamiltonian for Euler equations in the shallow water regime and thereby is shown to be bi-Hamiltonian and has a Lax pair formulation in [1]. Dullin-Gottwald-Holm equation (1.1) (DGH) combines the linear dispersion of Korteweg-de Vries (KdV) equation with the nonlinear/nonlocal dispersion of the Camassa-Holm (CH) equation, yet still preserves integrability via the inverse scattering transform (IST) method. This IST-integrable class of equations contains both the KdV equation and CH equation as limiting cases.

Using the notation \(m = \phi - \alpha^2 \phi_{xx}\), we rewrite the DGH equation as
\[ \phi_t - \alpha^2 \phi_{xx,t} + 2w\phi_x + 3\phi\phi_x + \gamma\phi_{xxx} = \alpha^2 (2\phi_x\phi_{xx} + \phi\phi_{xxx}). \]  
(1.2)

Recently, the authors [2-5] studied the well-posedness of Cauchy problem and the scattering problem for the DGH equation. Moreover, the issue of passing to the limit as the dispersive parameter tends to zero for the solution of DGH equation was investigated, and the scattering data of the scattering problem for the equation were explicitly expressed in [2]. Octavian G. Mustafa [6] investigated the low regularity conditions needed for the Cauchy problem of DGH equation via the semigroup approach of quasilinear hyperbolic equations of evolution and the viscosity method. Yue Liu [9] investigated the problems of the existence of global solutions and the formation of singularities for the DGH equation. In [10], Gui et al. studied the limit behaviour of the solutions to a class of nonlinear dispersive wave equations, which can be seen as some extension of DGH equation. In [11], Xiaolian Ai, Lixin Tian and Guilong Gui investigated the low regularity solutions to the generalized DGH equation and by using the modification method for the initial data, they obtained the energy estimate of the corresponding solutions, and then, presented a sufficient condition which guarantees the existence of the low regularity weak solutions.

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In this paper, the equation (1.1) has been investigated for the exact solutions and the sectionwise description of the paper is as follows. Section 2 is devoted to the outline of Lie group method to generate various symmetries of the DGH equation, and an optimal system comprising four basic vector fields is identified. Section 3 contains the reduced systems of ordinary differential equations and their exact solutions. Final section is for conclusion.

2 Lie symmetry group

Lie’s method [12, 13] of infinitesimal transformation groups which essentially reduces the number of independent variables in partial differential equation (PDE) and reduces the order of ordinary differential equation (ODE) has been widely used in equations of mathematical physics, some recent and important contributions are in [12-21]. The classical method in partial differential equation (PDE) and reduces the order of ordinary differential equation (ODE) has been widely used in equations of mathematical physics, some recent and important contributions are in [12-21]. The classical method for finding symmetry reductions of PDEs is the Lie group method of infinitesimal transformations and the associated determining equations are an over determined linear system. In this section, we will obtain the symmetry groups using the Lie’s classical method. The method mainly consists of following main steps:

1. Let us first consider the Lie group of point transformations

\[
\begin{align*}
\phi^* &= \phi + \epsilon \eta(x, t, \phi) + O(\epsilon^2) \\
x^* &= x + \epsilon \zeta(x, t, \phi) + O(\epsilon^2) \\
t^* &= t + \epsilon \tau(x, t, \phi) + O(\epsilon^2),
\end{align*}
\]

which leaves the system (1.2) invariant. In other words, the transformations are such that if \( \phi \) is a solution of system (1.2), then \( \phi^* \) is also a solution. The method for determining the symmetry group of (1.2) mainly consists of finding the infinitesimals \( \tau, \xi \) and \( \eta \), which are functions of \( x, t, \phi \).

2. Assuming that the system (1.2) is invariant under the transformations (2.1), we get the following relations from the invariance conditions (2.2), by setting the coefficients of different differentials equal to zero. We obtain a large number of determining equations for the group infinitesimals \( \tau, \xi, \) and \( \eta \), which we get from (2.2) after equating the coefficients of various derivative terms to zero, is as follows:

\[
\begin{align*}
\tau_x &= \tau_{\phi} = 0 \\
\xi_x &= \xi_{\phi} = 0 \\
\eta_x &= \eta_{\phi} = 0 \\
\alpha^2(\xi_t - \eta - \phi \tau_t) + \gamma \tau_t &= 0 \\
\alpha^2(\eta_{\phi} + 2\eta_x) &= 0 \\
-2\alpha^2 \eta_{xx} + 3\xi - \zeta + (3\phi + 2w)\tau_t &= 0 \\
\eta_{\phi} + \tau_t &= 0 \\
2w\eta_t - \alpha^2 \phi \eta_{xxx} + 3\phi \eta_x + \gamma \eta_{xx} - \alpha^2 \eta_{ext} + \eta_t &= 0.
\end{align*}
\]

3. The general solution of equations (2.3) provides following forms for the infinitesimal elements \( \eta, \xi \) and \( \tau \):

\[
\begin{align*}
\eta &= a(\phi + w + \frac{3\gamma}{2\alpha^2}) \\
\xi &= a(w + \frac{3\gamma}{2\alpha^2})t + c \\
\tau &= -at + b,
\end{align*}
\]

where \( a, b \) and \( c \) are arbitrary constants.

4. The Lie algebra associated with equation (1.2) consists of following three vector fields:

\[
\begin{align*}
V_1 &= -t \frac{\partial}{\partial t} + t(w + \frac{3\gamma}{2\alpha^2}) \frac{\partial}{\partial w} + (\phi + w + \frac{\gamma}{2\alpha^2}) \frac{\partial}{\partial \phi} \\
V_2 &= \frac{\partial}{\partial \phi} \\
V_3 &= \frac{\partial}{\partial \xi}.
\end{align*}
\]
In general, there are infinite number of subalgebras of this Lie algebra formed from any linear combination of generators $V_j, j = 1, 2, 3$ and to each subalgebra one can get the reduction using characteristic equations

$$\frac{d\phi}{\eta} = \frac{dx}{\xi} = \frac{dt}{\tau}. \quad (2.6)$$

We will work out first an optimal system and then embark upon the various reductions associated with generators in the optimal system. We begin by considering a general element $V = a_1V_1 + a_2V_2 + a_3V_3$ of symmetry algebra and subject it to various adjoint transformations to simplify it as much as possible (refer to [12]).

The adjoint action is given by the Lie series

$$Ad(\exp(\epsilon V_i))V_j = V_j - \epsilon[V_i, V_j] + \frac{\epsilon^2}{2}[V_i, [V_i, V_j]] - ..., \quad (2.7)$$

where $[V_i, V_j] = V_iV_j - V_jV_i$ is the commutator for the Lie algebra, and $\epsilon$ is a parameter.

The Commutator Table and the Adjoint Table for Lie algebra (2.5) are as follows:

<table>
<thead>
<tr>
<th>Table 1: Commutator Table</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_1$</td>
</tr>
<tr>
<td>$V_1$</td>
</tr>
<tr>
<td>$V_2$</td>
</tr>
<tr>
<td>$V_3$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2: Adjoint Table</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Ad$</td>
</tr>
<tr>
<td>$V_1$</td>
</tr>
<tr>
<td>$V_2$</td>
</tr>
<tr>
<td>$V_3$</td>
</tr>
</tbody>
</table>

The optimal system consists of the following four basic vector fields:

$$V_1 + \mu V_3$$
$$V_2 + \beta V_3$$
$$V_2$$
$$V_3. \quad (2.8)$$

The similarity variables and forms can be obtained by solving characteristic equations (2.6). The general solution of these equations involves two constants; one becomes the new independent variable $\xi$ and the other, say $F$, plays the role of new dependent variable. On substituting these solutions of (2.6) in equation (1.2), one gets the reduced ordinary differential equation.

3 Reduced ODEs and exact solutions

In this section, the primary focus is on the reductions associated with the vector fields (2.8) and attempt to find some exact solutions.

3.1 Vector field $V_1 + \mu V_3$

For this vector field, on using the characteristic equations (2.6), the similarity variable and the form of similarity solution is as follows:

$$\xi(t, x) = t^\mu \exp(x + (w + \frac{3\gamma^2}{2\alpha^2})t)$$
$$\phi(t, x) = \frac{1}{t}F(\xi) - (w + \frac{3\gamma^2}{2\alpha^2}), \quad (3.1)$$

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On using these in (1.2), the reduced ODE is given by
\[
-F(\xi) + (\mu + \alpha^2(1 - \mu))\xi F'\xi + (\alpha^2(1 - 3\mu))\xi^2 F''(\xi) - \alpha^2(2\alpha^3 F''(\xi) - 3\alpha^2\xi F'(\xi) F''(\xi) + \xi F(\xi) F'(\xi)(3 - \alpha^2) - 2\alpha^2\xi^2 F'(\xi) F''(\xi) - 2\alpha^2\xi^2(\xi^2 F'(\xi))^2 - \alpha^2\xi^3 F(\xi) F''(\xi) = 0,
\]
where prime (') denotes the differentiation with respect to the variable \(\xi\). On transforming the independent variable by the relation \(\xi = \exp(\zeta)\), the ODE (3.2) becomes
\[
-F + \mu \ddot{F} + \alpha^2 \dddot{F} - \alpha^2 \mu \dddot{F} + 3 \dot{F} \dddot{F} - 2 \alpha^2 \dddot{F} \dddot{F} = 0.
\]
where dot denotes the differentiation with respect to the variable \(\zeta\). We solved the reduced ODE (3.3) with the help of Maple and the following traveling wave solutions are obtained:

1. **Solutions in terms of \(\sin()\) function**

   \(\phi(t, x) = -w - \frac{\gamma}{2\alpha^2} + C_1 \sin(C_2 - \frac{\mu}{t} (\log t + x + (w + \frac{\gamma}{2\alpha^2}) t))\)

2. **Solutions in terms of \(\sinh()\) function**

   \(\phi(t, x) = -w - \frac{\gamma}{2\alpha^2} + C_1 \sinh(C_2 - \frac{\mu}{t} (\log t + x + (w + \frac{\gamma}{2\alpha^2}) t))\)

3. **Solutions in terms of \(\cos()\) function**

   \(\phi(t, x) = -w - \frac{\gamma}{2\alpha^2} + C_1 \cos(C_2 - \frac{\mu}{t} (\log t + x + (w + \frac{\gamma}{2\alpha^2}) t))\)

4. **Solutions in terms of \(\cosh()\) function**

   \(\phi(t, x) = -w - \frac{\gamma}{2\alpha^2} + C_1 \cosh(C_2 - \frac{\mu}{t} (\log t + x + (w + \frac{\gamma}{2\alpha^2}) t))\)

where \(C_1\) and \(C_2\) are arbitrary constants.

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where $C_1$ and $C_2$ are arbitrary constants.

5. **Solutions in terms of \( \exp() \) function**

\[
\begin{align*}
(i) \quad & \phi(t, x) = -w - \frac{\gamma}{\alpha^2} + C_1 \exp(C_2 - \frac{1}{\gamma} (\log x + (w + \frac{\gamma}{\alpha^2} t))) + C_2 \\
(ii) \quad & \phi(t, x) = -w - \frac{\gamma}{\alpha^2} + C_1 \exp(C_2 + \frac{1}{\gamma} (\log x + (w + \frac{\gamma}{\alpha^2} t))) \\
(iii) \quad & \phi(t, x) = -w - \frac{2\gamma}{\alpha^2} + C_1 (\exp(C_2 - \frac{1}{\gamma} (\log x + (w + \frac{\gamma}{\alpha^2} t)))^2 \\
(iv) \quad & \phi(t, x) = -w - \frac{2\gamma}{\alpha^2} + C_1 (\exp(C_2 + \frac{1}{\gamma} (\log x + (w + \frac{\gamma}{\alpha^2} t)))^2
\end{align*}
\]

(3.8)

where $C_1$ and $C_2$ are arbitrary constants.

### 3.2 Vector field $V_2 + \beta V_3$

For this vector field, the form of the similarity variable and similarity solution is as follows:

\[
\begin{align*}
\xi(t, x) &= \beta t - x \\
\phi(t, x) &= F(\xi)
\end{align*}
\]

(3.9)

The reduced ODE in this case is as follows:

\[
\beta F'(\xi) - \alpha^2 \beta F''(\xi) - 2wF'(\xi) - 3F(\xi)F'(\xi) - \gamma F''(\xi) + \alpha^2 (2F'(\xi)F''(\xi) + F(\xi)F''(\xi)) = 0,
\]

(3.10)

where prime(\(')\) denotes the differentiation with respect to the variable \(\xi\). On solving the above ODE, we get the following traveling wave solutions:

1. **Solutions in terms of \( \tanh() \) function**

\[
\begin{align*}
(i) \quad & \phi(t, x) = -w - \frac{\gamma}{\alpha^2} + \frac{\alpha^2 \gamma w + \gamma}{\alpha^2 (-1 + \tanh (\frac{\alpha^2 \gamma w + \gamma}{\alpha^2 \gamma w} ))^2} \\
(ii) \quad & \phi(t, x) = -w - \frac{\gamma}{\alpha^2} + \frac{\alpha^2 \gamma w + \gamma}{\alpha^2 (-1 + \tanh (\frac{\alpha^2 \gamma w + \gamma}{\alpha^2 \gamma w} ))^2} \\
(iii) \quad & \phi(t, x) = constant,
\end{align*}
\]

(3.11)

where $C_1$ is arbitrary constant.

### 3.3 Vector field $V_2$

In this case, the form of the similarity variable and similarity solution is as follows:

\[
\begin{align*}
\xi(t, x) &= x \\
\phi(t, x) &= F(x)
\end{align*}
\]

(3.12)

The reduced ODE in this case is as follows:

\[
2wF'(x) + 3F(x)F'(x) - 2\alpha^2 F'(x)F''(x) - \alpha^2 F(x)F''(x) + \gamma F''(x) = 0,
\]

(3.13)

where prime(\(')\) denotes the differentiation with respect to the variable \(x\). We solved the reduced ODE (3.13) with the help of Maple and the following traveling wave solutions are obtained:

1. **Solutions in terms of \( \tanh() \) function**

\[
\begin{align*}
(i) \quad & \phi(t, x) = \frac{-(\gamma + 2\alpha^2 w)}{\alpha^2 (1 + \tanh (\frac{\alpha^2 \gamma w + \gamma}{\alpha^2 \gamma w} ))} \\
(ii) \quad & \phi(t, x) = \frac{-(\gamma + 2\alpha^2 w)}{\alpha^2 (-1 + \tanh (\frac{\alpha^2 \gamma w + \gamma}{\alpha^2 \gamma w} ))} \\
(iii) \quad & \phi(t, x) = \frac{\alpha^2 (\gamma + 2\alpha^2 w)}{\alpha^2 (1 + \tanh (\frac{\alpha^2 \gamma w + \gamma}{\alpha^2 \gamma w} ))} \\
(iv) \quad & \phi(t, x) = \frac{\alpha^2 (\gamma + 2\alpha^2 w)}{\alpha^2 (-1 + \tanh (\frac{\alpha^2 \gamma w + \gamma}{\alpha^2 \gamma w} ))} \\
(v) \quad & \phi(t, x) = \frac{\alpha^2 \gamma w + \gamma + 2\alpha^2 w}{\alpha^2 (1 + \tanh (\frac{\alpha^2 \gamma w + \gamma}{\alpha^2 \gamma w} ))} \\
(vi) \quad & \phi(t, x) = \frac{\alpha^2 \gamma w + \gamma + 2\alpha^2 w}{\alpha^2 (-1 + \tanh (\frac{\alpha^2 \gamma w + \gamma}{\alpha^2 \gamma w} ))} \\
(vii) \quad & \phi(t, x) = constant,
\end{align*}
\]

(3.14)

where $C_1$ is arbitrary constant.
3.4 Vector field $V_3$

For this vector field, the similarity variable and the form of similarity solution is as follows:

$$\xi(t, x) = t$$
$$\phi(t, x) = F(t), \quad (3.15)$$

On using these in equation (1.2), the reduced ODE is given by

$$F'(t) = 0, \quad (3.16)$$

where prime (') denotes the differentiation with respect to the variable $t$. We solved the reduced ODE (3.16) and got the following solution:

$$\phi(t, x) = \text{constant}. \quad (3.17)$$

4 Conclusion

We have investigated the symmetries of the Dullin-Gottwald-Holm equation. The Lie group method is utilized for the purpose of obtaining the group infinitesimals. The basic fields of the optimal system lead to reductions to Ordinary Differential Equations and for each element in the optimal system, a number of traveling wave solutions are obtained. The availability of mathematical computer software like Maple facilitates the tedious algebraic calculations. It is worth to mention here that the correctness of the solutions has been checked with the aid of software Maple.

References


