Exact Solutions to the Dispersive Long Wave Equation

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Abstract: Based upon the extended projective Riccati equations method, we investigate the dispersive long wave equation (DLWE) using the symbolic computation system. Several families of analytical solutions are obtained, including some new and more general ones.

Key words: Extended projective Riccati equations method; Exact solutions; Dispersive long wave equations (DLWE)

1 Introduction

The (2+1)-dimensional dispersive long wave equation (DLWE)

\[
\begin{align*}
    u_y + v_{xx} + u_xu_y + uu_{xy} &= 0, \\
    v_t + u_x + uu_x + u_{xx} &= 0,
\end{align*}
\]

(1)

was first obtained by Boiti as a compatibility condition for a weak Lax pairs[1]. There are lots of papers discussing its possible applications and exact solutions. In [2], Paquin and Winternitz showed that the symmetry algebra of DLWE possesses the infinite-dimensional Kac-Moody-Virasoro structure. Some special similarity solutions are given in [3] by using symmetry method and the classical group analysis. The more general symmetry algebra, \( W_\infty \) symmetry algebra, is given in [4]. Recently, Fan et al. obtained some exact solutions of the equation by the ansatz-based method, including traveling-wave solutions, multiple-soliton solutions, soliton-like solutions, periodic solutions and Weierstrass function solutions[5-9]. In this paper, we aim to seek more general analytical solutions of the dispersive long wave equation by the extended projective Riccati equations method.

2 Summary of the extended method

The key idea of the extended projective Riccati equations method is to take full advantages of the following projective system[10]

\[
\begin{align*}
    f'(\xi) &= pf(\xi)g(\xi), \\
    g'(\xi) &= 1 + pg^2(\xi) - rf(\xi)
\end{align*}
\]

(2)

which has a first integral

\[
g^2(\xi) = -\frac{1}{p}[1 - 2rf(\xi) + (r^2 + \delta)f^2(\xi)],
\]

(3)

where \( p = \pm 1 \), \( r \) is a real constant and \( \delta \) is also a real constant which depends on the concrete expressions of \( f(\xi), g(\xi) \). On the basis of [11], we can get following solutions to this system:

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When \( p = -1 \),

\[
    f(\xi) = \frac{1}{r + k \sinh(\xi) + l \cosh(\xi)}, \quad g(\xi) = \frac{kcosh(\xi) + l \sinh(\xi)}{r + k \sinh(\xi) + l \cosh(\xi)},
\]

where \( k \) and \( l \) are real constants.

When \( p = 1 \),

\[
    f(\xi) = \frac{1}{r + m \sin(\xi) + n \cos(\xi)}, \quad g(\xi) = -\frac{mcos(\xi) - n \sin(\xi)}{r + m \sin(\xi) + n \cos(\xi)},
\]

where \( m \) and \( n \) are real constants.

Now we establish the extended method as follows. Suppose we are given a partial differential equation for a function \( u(x, y, t) \):

\[
    H(u, u_t, u_x, u_y, u_{xx}, u_{xy}, u_{xt}, u_{yt}) = 0.
\]

**Step 1.** We assume that (6) has the following solutions:

\[
    u(x, y, t) = a_0 + \sum_{i=1}^{N} [a_i f^i(\xi) + b_i f^{i-1}(\xi) g(\xi)],
\]

where \( a_0, a_i, b_i, \xi \) are all unknown functions of \( x, y, t \), \( f(\xi) \) and \( g(\xi) \) satisfy (2). The parameter \( N \) can be found by balancing the highest order derivative term and the nonlinear terms in (6).

**Step 2.** Substituting (7) along with (2,3) into (6) yields a set of algebraic polynomials for \( f^i(\xi)g^j(\xi) (i = 0, 1, ..., j = 0, 1) \). Setting the coefficients of these terms \( f^i(\xi)g^j(\xi) \) to zero, we’ll get a system of over-determined PDEs with respect to unknown functions \( a_0, a_i, b_i \) and \( \xi \).

**Step 3.** Solving the above system by using the symbolic computation system Maple, we would end up with the explicit expressions for \( a_0, a_i, b_i \) and \( \xi \) or the constraints among them. Sometimes in order to get analytical results we need to make a prior ansatz.

**Step 4.** According to the solutions (4) and (5) of (2) and the results in last step, we can obtain many families of exact solutions for the given PDE.

### 3 Exact solutions to DLWE

By balancing the highest-order contributions from both the linear and nonlinear terms in (1), we can assume the solutions in the form:

\[
    \begin{cases}
        u(x, y, t) = a_0 + a_1 f(\xi) + b_1 g(\xi), \\
        v(x, y, t) = A_0 + A_1 f(\xi) + B_1 g(\xi) + A_2 f^2(\xi) + B_2 f(\xi) g(\xi),
    \end{cases}
\]

where \( a_0 = a_0(y, t), a_1 = a_1(y, t), b_1 = b_1(y, t), A_0 = A_0(y, t), A_1 = A_1(y, t), A_2 = A_2(y, t), B_1 = B_1(y, t), B_2 = B_2(y, t) \) and \( \xi = \pm \alpha(y, t) + \beta(y, t) \) are all differential functions. The aim of choosing these functions to be special forms, i.e., the \( x \) independence of \( a_0, a_1 \) etc., is to make calculation feasible.

Substituting (8) along with (2,3) into (1), collecting coefficients of monomials of \( f(\xi), g(\xi), x \) of the resulting system’s numerator(Notice that \( a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2, \alpha, \beta \) are independent of \( x \)), then setting each coefficient to zero, we obtain an over-determined PDE system. Solving the PDE system by means of Maple gives the explicit expressions of the unknowns. Then we have the following exact solutions to EQ.1:(Note: In the rest of this paper, \( F_i (i = 1, 2, 3, 4, 5) \) are arbitrary functions with respect to corresponding independent variables, and \( \frac{d}{dy} \) denotes \( \frac{d}{dy} \)).

**Case 1.** When \( p = -1 \) we have:

**subcase 1.1:**

\[
    \begin{cases}
        u = F_1(t) \pm 2 \frac{\sqrt{r^2 + \delta} F_2(t)}{r + l \cosh(\xi) + k \sinh(\xi)}, \\
        v = F_3(y) - \frac{2r (r^2 + \delta) [1 + F_3(y)]}{\delta [r + l \cosh(\xi) + k \sinh(\xi)]} + 2 \frac{(r^2 + \delta)^2 [1 + F_3(y)]}{\delta [r + l \cosh(\xi) + k \sinh(\xi)]^2},
    \end{cases}
\]
where $\xi = F_2(t)x - \frac{r^2+\delta}{\delta}\int_{F_2(t)}^{1+F_3(y)}dy + F_4(t)$ and $\delta = k^2 - l^2$.

**subcase 1.2:**

$$
\begin{align*}
    u &= F_1(t) \mp \frac{F_3(t) \sqrt{r^2+\delta}}{r + l \cosh(\xi) + k \sinh(\xi)} + \frac{F_3(t) [l \sinh(\xi) + k \cosh(\xi)]}{r + l \cosh(\xi) + k \sinh(\xi)}, \\
    v &= -1 \pm \frac{F_2(y) r}{\sqrt{r^2+\delta} [r + l \cosh(\xi) + k \sinh(\xi)]} \mp \frac{F_2(y) (r^2 + \delta)}{\sqrt{r^2+\delta} [r + l \cosh(\xi) + k \sinh(\xi)]^2}.
\end{align*}
$$

where $\xi = -F_3(t)x + \int F_3(t) F_1(t) dt + F_4(y)$ and $\delta = k^2 - l^2$.

**subcase 1.3:**

$$
\begin{align*}
    u &= F_2(y) + F_1(t) \mp \frac{\sqrt{r^2+\delta} F_3(t)}{r + l \cosh(\xi) + k \sinh(\xi)} + \frac{F_3(t) [l \sinh(\xi) + k \cosh(\xi)]}{r + l \cosh(\xi) + k \sinh(\xi)}, \\
    v &= -1 - F_2'(y) \mp \frac{[\int \sqrt{r^2+\delta} F_3^2(t) F_1'(y) dt + F_5(y)] r}{\sqrt{r^2+\delta} [r + l \cosh(\xi) + k \sinh(\xi)]} \\
    &\quad \pm \frac{[\int \sqrt{r^2+\delta} F_3^2(t) F_1'(y) dt + F_5(y)] [\sqrt{r^2+\delta}]}{[r + l \cosh(\xi) + k \sinh(\xi)]^2} \\
    &\quad + \frac{\int \sqrt{r^2+\delta} F_3^2(t) F_1'(y) dt + F_5(y)] [l \sinh(\xi) + k \cosh(\xi)]}{[r + l \cosh(\xi) + k \sinh(\xi)]^2},
\end{align*}
$$

where $\xi = -F_3(t)x + \int F_3(t) [F_2(y) + F_1(t)] dt + F_4(y)$ and $\delta = k^2 - l^2$.

**Case 2.** When $p = 1$ we have:

**subcase 2.1:**

$$
\begin{align*}
    u &= F_1(t) \pm 2 \frac{\sqrt{-r^2 - \delta} F_2(t)}{r + n \cos(\xi) + m \sin(\xi)}, \\
    v &= F_3(y) - 2 \frac{r (r^2 + \delta)}{\delta [r + l \cosh(\xi) + k \sinh(\xi)]} \mp 2 \frac{(r^2 + \delta)^2 [1 + F_3(y)]}{\delta [r + l \cosh(\xi) + k \sinh(\xi)]^2},
\end{align*}
$$

where $\xi = F_2(t)x + \frac{r^2+\delta}{\delta}\int_{F_2(t)}^{1+F_3(y)}dy + F_4(t)$ and $\delta = -m^2 - n^2$.

**subcase 2.2:**

$$
\begin{align*}
    u &= F_1(t) \pm \frac{F_3(t) \sqrt{-r^2 - \delta}}{r + n \cos(\xi) + m \sin(\xi)} + \frac{F_3(t) [n \sin(\xi) - m \cos(\xi)]}{r + n \cos(\xi) + m \sin(\xi)}, \\
    v &= -1 \pm \frac{F_2(y) r}{\sqrt{-\delta - r^2} [r + n \cos(\xi) + m \sin(\xi)]} \mp \frac{F_2(y) (r^2 + \delta)}{\sqrt{-\delta - r^2} [r + n \cos(\xi) + m \sin(\xi)]^2}.
\end{align*}
$$

where $\xi = F_3(t)x - \int F_3(t) F_1(t) dt + F_4(y)$ and $\delta = -m^2 - n^2$.

**subcase 2.3:**

$$
\begin{align*}
    u &= F_2(y) + F_1(t) \mp \frac{\sqrt{-r^2 - \delta} F_3(t)}{r + n \cos(\xi) + m \sin(\xi)} + \frac{F_3(t) [n \sin(\xi) - m \cos(\xi)]}{r + n \cos(\xi) + m \sin(\xi)}, \\
    v &= -1 - F_2'(y) \mp \frac{[\int \sqrt{-r^2 - \delta} F_3^2(t) F_1'(y) dt + F_5(y)] r}{\sqrt{-r^2 - \delta} [r + n \cos(\xi) + m \sin(\xi)]} \\
    &\quad \pm \frac{[\int \sqrt{-r^2 - \delta} F_3^2(t) F_1'(y) dt + F_5(y)] (r^2 + \delta)}{\sqrt{-r^2 - \delta} [r + n \cos(\xi) + m \sin(\xi)]^2} \\
    &\quad + \frac{[\int \sqrt{-r^2 - \delta} F_3^2(t) F_1'(y) dt + F_5(y)] [n \sin(\xi) - m \cos(\xi)]}{[r + n \cos(\xi) + m \sin(\xi)]^2},
\end{align*}
$$

where $\xi = F_3(t)x - \int [F_2(y) + F_1(t)] F_3(t) dt + F_4(y)$ and $\delta = -m^2 - n^2$.
4 Discussion

In this paper, various exact solutions of the dispersive long wave equation are derived with the aid of the coupled projective Riccati equations and symbolic computation. We hope that the approach taken in our present work may be further extended to other nonlinear systems.

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References


