Optimal Control of Kuramoto-Sivashing Equation

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Abstract. The paper is to consider optimal control of Kuramoto-Sivashing equation with boundary condition. The problem of optimal control of Kuramoto-Sivashing equation is presented firstly. And then the existence of optimal control of the equation with boundary condition is proved.

Keywords: Kuramoto-Sivashing equation ; optimal control

1 Introduction

Optimal control is an important component of modern control theory. Kuramoto-Sivashing equation is a typical nonlinear evolution equation. Many previous studies dwelted on the research of Kuramoto-Sivashing equation, but few on optimal control of Kuramoto-Sivashing equation, which calls for further study. The feedback control is given by A.Armaou and P.D.Christofides[1]. Changbing Hu, Roger Temam make a research of robust control of Kuramoto-Sivashing equation[2].The Optimal control of stronger Burgers equation is given by Zhifeng Zhao, Lixin Tian[3].

On the basis of previous research, we discuss the problem of optimal control of Kuramoto-Sivashing equation. The paper is organized as follows: some notations, lemmas and theorem are given in section 2. The existence of optimal control of Kuramoto-Sivashing equation is proved in section 3. In section 4, the conclusion is drawn.

2 Notations and Lemmas

For $T > 0$, suppose $Q = (0, T) \times \Omega$, where $\Omega = (0, 1)$. Let $V = H_0^1(0, 1)$ and $V^*$ be the dual space of $V$. $H = L^2(0, 1)$ denotes the space of measurable functions which are square integrable. The space $W(0, T; V)$ is define by $W = \{ \varphi \in L^2(\Omega) \cap H_0^1(\Omega) | \varphi_t \in L^2(\Omega) \cap H_0^1(\Omega) \}$. $u(t), f(t)$, denote $u(\cdot, t), f(\cdot, t)$, respectively. Define the following inner space on $V$ : $\langle \varphi, \psi \rangle_V = \langle \varphi', \psi' \rangle_H$.

Consider the following Kuramoto-Sivashing equation

$$u_t - ku_{xx} + u_{xxx} + uu_x = f,$$  \hspace{1cm} (2.1)

$$u(x, 0) = \phi(x),$$  \hspace{1cm} (2.2)

$$u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = 0,$$  \hspace{1cm} (2.3)

where $f \in L^2(V^*), \phi \in H$.

Let $b(\varphi, \psi, \phi) = \frac{1}{3} \int_0^1 [\varphi' \psi' \phi + \varphi \psi' \phi'] dx, \forall \varphi, \psi, \phi \in H_0^1(0, 1)$.

Definition 1 A function $u(x, t) \in W(V)$ is called a weak solution to eq.(1) if for all $\varphi \in V$ and $u(0) = \phi$,

$$\langle u_t(t), \varphi \rangle_V - k \langle u_{xx}(t), \varphi \rangle_V + \langle u_{xxx}(t), \varphi \rangle_V + b(u(t), u(t), \varphi) = \langle f(t), \varphi \rangle_{V^*, V}$$  \hspace{1cm} (2.4)

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holds.

**Remark 1** For all $\phi \in H$, there exits a weak solution $u(x, t) \in W(V)$ to eq.(1).

Proof: See [4],[5]. In order to prove the theorem in Section 3, we need the following lemmas.

**Lemma 1** For all $u \in V$ and $1 \leq p \leq \infty$, we have $\|u\|_{L^p} \leq \|u\|_V$.

Proof: Let $u \in \{\phi | \phi \in C^1[0,1], \phi(0) = 0\}$, then we infer from Hölder’s inequality that

$$|u(x)| \leq \int_0^x |u'| \, dt \leq \left( \int_0^1 |u'|^2 \, dt \right)^{\frac{1}{2}} = \|u\|_H.$$  

For all $1 \leq p \leq \infty$, it is easy to see that $\|u\|_{L^p} \leq (\int_0^1 |u|^p \, dx)^{\frac{1}{p}} \leq (\int_0^1 |u|_{H}^p \, dx)^{\frac{1}{p}} = \|u\|_V$ and $\|u\|_{L^\infty} = \text{ess sup}_{x \in (0,1)} |u(x)| \leq \|u\|_V$. Since $\{\phi | \phi \in C^1[0,1], \phi(0) = 0\}$ is dense in $V$, lemma 1 holds for arbitrary $u \in V$ and $1 \leq p \leq \infty$.

**Lemma 2** Let $\phi \in H, f \in L^2(u^*)$. There exists a constant $M_0$ such that

$$\|u\|_{W(V)} \leq M_0[1 + (\|\phi\|_H + \|f\|_{L^2(u^*)})^2] \quad \text{(2.5)}$$

Proof: we first prove that

$$\|u\|_{L^2}^2 \leq M_1(\|\phi\|_H + \|f\|_{L^2(u^*)})^2 \quad \text{(2.6)}$$

holds for a constant $M_1$. For this purpose, multiply $u$ on both sides of eq.(1) and integrate equation over the interval $(0, 1)$. This leads to

$$\frac{1}{2} \int_0^T (\|f(t)\|_{V^*}^2 + \|u(t)\|_{V^*}^2 + \|u(t)\|_{H}^2) \, dt = \langle f(t), u(t) \rangle_{V^*, V} \quad \text{(2.7)}$$

From Hölder’s inequality, we have

$$\int_0^T \langle f(t), u(t) \rangle_{V^*, V} \, dt \leq \int_0^T \|f(t)\|_{V^*} \|u(t)\|_{V} \, dt \leq \|f(t)\|_{L^2(u^*)} \|u(t)\|_{L^2(V)} \quad \text{(2.8)}$$

Integrating (7) by $t$ over $(0, T)$, we obtain

$$\frac{1}{2} \|u(T)\|_{H}^2 + k \|u\|_{L^2(V)}^2 + \int_0^T \|u_{xx}\|_{H}^2 \, dt \leq \frac{1}{2} \|\phi\|_{H}^2 + \|f\|_{L^2(u^*)} \|u\|_{L^2(V)} \quad \text{(2.9)}$$

According to Lemma 1 and Young inequality, the following inequality

$$2(k + 1) \|u\|_{L^2(V)}^2 \leq \|\phi\|^2 + \frac{1}{k} \|f\|_{L^2(u^*)}^2 + k \|u\|_{L^2(V)}^2 \quad \text{(2.10)}$$

holds. Hence, we have

$$\|u\|_{L^2(V)}^2 \leq \frac{1}{k + 2} \|\phi\|^2 + \frac{1}{k(k + 2)} \|f\|_{L^2(V)}^2 \quad \text{(2.11)}$$

Take $M_1 = \max \{1/k + 2, 1/(k + 2)\}$, such that $\|u\|_{L^2}^2 \leq M_1 \left(\|\phi\|_H + \|f\|_{L^2(u^*)}\right)^2$, then (6) holds.

In the following part we prove that $\|\phi\|_H$ and $\|f\|_{L^2(u^*)}$ are bounded. Due to eq.(7), we have

$$(2T + 1) \|u\|_{H}^2 \leq \|\phi\|_{H}^2 + 2 \|f\|_{L^2(V)} \|u\|_{L^2(V)}^2 \leq \|\phi\|_{H}^2 + 2 \|f\|_{L^2(u^*)} \sqrt{M_1} \left(\|\phi\|_H + \|f\|_{L^2(u^*)}\right) \quad \text{(2.12)}$$

Hence, we have

$$\|u\|_{C(H)}^2 \leq M_2(\|\phi\|_H + \|f\|_{L^2(u^*)}^2), \quad \text{(2.13)}$$

where $M_2 = \max \{1, 2\sqrt{M_1}\}/2T$.

By eq.(1), we have

$$\|u(t)\|_{V^*} \leq \|f\|_{V^*} + k \|u(t)\|_{V} + \|u(t)\|_{H} \|u(t)\|_{V} + \|u_{xx}(t)\|_{H}$$

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3 Optimal control of Kuramoto-Sivashing equation

In this section, we study the following Kuramoto-Sivashing equation with control term in this part.

\[ u_t - ku_{xx} + u_{xxxx} + uu_x = f + m \]  \hspace{1cm} (3.15)

\[ u(x,0) = \phi(x) \]  \hspace{1cm} (3.16)

\[ u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = 0 \]  \hspace{1cm} (3.17)

where \( \phi \in H \).

For a control term \( m \in L^2(W) \), there exists a weak solution to eqs.(15)-(17) by theorem 1.

Now we consider the following cost function

\[ J(u, m) = \frac{1}{2} \| u - z \| + \frac{\sigma}{2} \| m \|^2 \]  \hspace{1cm} (3.18)

where \( Z \) is a desired state and \( \sigma > 0 \) is fixed. According to the control theory, the above control belongs to feedback control. From another point, the same control problem belongs to inverse problem of fluid Mechanics. In fact, if we know the control term through the equipments, we can make good use of the input term \( m \). The optimal control problem is

\[ \min J(u, m) \]  \hspace{1cm} (3.19)

Where \( (u, m) \) solves eq.(15). Let \( X = W(V) \times L^2(Q) \) and \( Y = L^2(V) \times H \).

We introduce the operator \( e = (e_1, e_2) : X \to \Lambda \) which is defined by

\[ e(u, m) = \left[ (\Delta)^{-1} (u_t - ku_{xx} + u_{xxxx} + uu_x - f - m), u(\cdot, 0) - \phi \right] \]

where \( \Delta \) is Laplace operator from \( H^1_0(\Omega) \) to \( H^{-1}(\Omega) \). So problem (19) can be translated into

\[ \min J(u, m) \quad s.t. \quad e(u, m) = 0 \]  \hspace{1cm} (3.20)

**Theorem 1** There exists an optimal control solution \((u^*, m^*)\) to problem (20) where \( u^* \) and \( m^* \) satisfies eqs.(15)-(17).

**Proof:** Let \( (u, m) \in X \) satisfying \( e(u, m) = 0 \), then \( J(u, m) \geq \frac{\sigma}{2} \| m \|^2_{L^2(W)} \). By Lemma 2, we have \( \| u \|_{W(V)} \to \infty \) as \( \| m \|_{L^2(W)} \to \infty \). When \( \| (u, m) \|_X \to \infty \), we obtain

\[ J(u, m) \to \infty \]  \hspace{1cm} (3.21)

and

\[ \lim_{n \to \infty} \inf J(u^n, m^n) \geq \frac{1}{2} \| cu^* \|^2 - \| (cu^* - Z) \| + \frac{1}{2} \| Z \|^2 + \frac{\sigma}{2} \| u^* \|^2_{L^2} = J(u^*, m^*) \]  \hspace{1cm} (3.22)

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So $J$ is a weak continuous unbounded function on $X$. For arbitrary $(u, m) \in X$, we have $J(u, m) > 0$. Then there is $\zeta \geq 0$ satisfying
\[ \zeta = \inf\{ J(u, m) : (u, m) \in X \text{ and } e(u, m) = 0 \} \] (3.23)
This implies that there exists a minimizing serial $\{ (u_k, m_k) \}_{k \in N}$ on $X$ which satisfies that for $\forall k \in N$, $\zeta = \lim_{k \to \infty} J(u_k, m_k), e(u_k, m_k) = 0$. Due to (21), there exists $(u^*, m^*) \in X$ satisfying
\[ u_k \to u^*, k \to \infty, u \in W(V) \] (3.24)
\[ m_k \to m^*, k \to \infty, m \in L^2(W) \] (3.25)
We infer from (24) that for each $\phi \in L^2(V)$, there is
\[ \lim_{k \to \infty} \int_0^T (u_{ik}(t) - u_i^*(t)) V^* \varphi dt = 0 \] (3.26)
Hence $W(V) \to L^2(L^\infty)$, and then $u^n \to u^*$. As the serial $\{ u_k \}_{k \in N}$ weakly converges, $\| u_k \|_{W(V)}$ and $\| u_k \|_{C(H)}$ are all bounded. Thus, it follows from Höld inequality that for each $\varphi \in L^2(V)$
\[ \int_0^T \int_0^1 (u^n u^*_n - u^* u^*_n) \varphi dx dt = \frac{1}{2} \int_0^T \int_0^1 (u^* u^* - u^n u^n) \varphi dx dt \leq \frac{1}{2} \int_0^T \int_0^1 (u^* - u^n) u^n \varphi dx dt \]
\[ + \frac{1}{2} \left( \int_0^T \int_0^1 (u^* - u^n) u^* \varphi dx dt \right) \leq \frac{1}{2} \int_0^T \int_0^1 (u^* - u^n) u^n \varphi dx dt \]
\[ + \frac{1}{2} \left( \int_0^T \| u^* - u^n \|_{L^\infty} \| u^*(t) \|_{L^1} \| \varphi(t) \|_{H^1(\Omega)} dt \right) \leq \frac{1}{2} \| u^* - u^n \|_{L^2(L^\infty)} + \| u^n \|_{C(H)} \| \varphi \|_{L^2(V)} \]
(3.27)
Due to (25), we also have
\[ \int_0^T \int_0^1 (m_k - m^*) \varphi dx dt \to 0 (n \to \infty) \] (3.28)
then we conclude that $e(u^*, m^*) = 0$. Since $u^* \in W(V)$, we have $u^*(0) \in H$ in $W(V)$. From $u_k \to u^*$, we infer that $u_k(0) \to u^*(0)$ and $(u_k(0) - u^*(0), \varphi)_H \to 0 (k \to \infty)$. Hence $e(u^*(0), m^*(0)) = 0$.

4 Conclusion

In the paper, the optimal control problem of Kuramoto-Sivashing equation with boundary condition is presented. The existence of optimal controllability of the equation is proved. The work provides a theoretical basis for further study and application in engineering field.

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