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Bursting Oscillations as well as the Bifurcation Mechanism of a Four-dimensional Laser System with Different Excitation Frequency Ratios

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Abstract: Due to the complexity of the multi-scale effects in a nonlinear system under multiple excitation couplings, a non-smooth system with parametric and external excitations is explored. Firstly, by employing the Moivre formula, two excitation terms could be expressed by one slow-varying parameter. The obvious fast-slow effects in such systems could be presented when strictly adjusting both frequencies, making them much smaller than the natural frequency of the system. Furthermore, the stability of equilibria and possible bifurcations in fast subsystem are easily judged with the help of fast and slow analysis method. Notably, the frequency ratio between two excitations is the focus of the study. It is found that the number of fold and Hopf bifurcation points significantly increases and the structure of equilibrium curves becomes more complicated as the ratio increases, so that the trajectory also exhibits a variety of mixed-mode oscillations. The oscillations of the system also perform two different symmetrical forms according to the two cases where the frequency ratio is integer or fractional.

Keywords: multiple timescales; bursting oscillations; parametric and external excitations

1 Introduction

M. Faraday firstly discovered the phenomenon of parametric vibrations, which is achieved indirectly by changing the periodicity of the model parameters, mainly by external disturbing forces. Many motion models in engineering usually employ parametric excitations to describe the nonlinear problems, such as light pressure and gravitational uptake due to the periodic motion of the Sun, vibration problems of flexible beams in spacecraft, vibration and control problems of tethered satellites, etc.[1-3]. Therefore, it has great guidance significance on the study of dynamic behaviors about nonlinear systems with parametric excitations in the field of practical engineering.

Initially, most researches on multi-timescale coupled systems were carried out for only considering a single excitation term[4-5], but in fact, real engineering applications tend to contain the combination of multiple excitation terms[6-7]. The mathematical models constructed based on above factors usually show richer bursting oscillation phenomena. In addition, non-smooth factors are important aspects that cannot be ignored in actual engineering projects[8-9]. As the deep explosions by scholars, complicated dynamic behaviors of non-smooth systems with multiple exciting terms, for example, the fast-slow coupling effects arising from the system with two external excitations or a combination of external and parametric excitations, have gradually become hot topics, and remarkable research results are also obtained[11-12]. It could be revealed that excitation amplitude and frequency both generate great influences on bursting behaviors of non-autonomous systems. The study on frequency ratio has great value when two excitation terms coexist in the system.

Taking the above discussions into consideration, based on a four-dimensional laser system[12], a new system is established by introducing a periodic parametric excitation and an external excitation. Firstly, two excitations are transformed into a form expressed by the same slow variable with the help of Moivre formula. Therefore, the fast-slow coupling effects on the system under two frequency excitations are explored by traditional fast-slow analysis method[13-14]. Finally, various dynamical behaviors derived by the frequency ratio are emphasized.

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2 Mathematical model

In order to explore the bursting oscillations of a Filippov system with two excitation terms, the mathematical model was established by adding a piecewise control term, an external and a parametric excitation term on the basis of a four-dimensional laser system. The model could be expressed as the following

$$\begin{cases} \dot{x_1} = \sigma(x_2 - x_1) + \varepsilon sgn(x_1), \\ \dot{x_2} = -x_2 - \delta x_3 + (r - x_4 + A_1 cos(\Omega_1 t))x_1 + A_2 \cos(\Omega_2 t), \\ \dot{x_3} = \delta x_2 - x_3, \\ \dot{x_4} = -bx_4 + x_1 x_2, \end{cases} \tag{1}$$

where $x_i(i = 1, 2, 3, 4)$ are state variables, parameters σ, δ, r, b are real, ε is constant, $sgn(x_1)$ is a symbolic function, $A_1cos(\Omega_1 t)$ is the parametric exciting term of system (1) and $A_2cos(\Omega_2 t)$ is the external term, where $A_i(i = 1, 2)$ and $\Omega_i(i = 1, 2)$ represent exciting amplitudes and frequencies, respectively. While two exciting frequencies $\Omega_i(i = 1, 2)$ are both much smaller than the natural frequency ω_N of the system, i.e. $1 < \Omega_i(i = 1, 2) \ll 1$, system (1) will exhibit multiple time-scale, representing the effects of fast-slow coupling behaviors.

3 Theoretical analysis

Since there exists an order gap between two exciting frequencies and the natural one, two excitation terms could be treated as two slowly-changing parameters $w_1 = A_1cos(\Omega_1 t), w_2 = A_2cos(\Omega_2 t)$ of system (1). And then the system could be regarded as a generalized autonomous system with two slow variables, whose corresponding equilibrium points are called generalized equilibrium points. Moreover, if there is a certain ratio relationship between two frequencies directly or indirectly, then an elementary slow-variable function can be found to represent these two slow variables simultaneously, transforming the original system into a fast-slow coupled system containing only one slow variable, thus studying the mixed-mode oscillations of the system via the traditional fast-slow analysis. Here, Moivre formula is introduced to perform the transformation, the expression is defined as

$$(cosx + isinx)^n = cons(nx) + isin(nx), \tag{2}$$

where n denotes an arbitrary positive integer, i denotes a pure imaginary number ($i^2 = -1$). Expanding the left side of Eq. (2) and integrating the real and imaginary parts, the expression of $cos(nx)$ can be obtained, shown as

$$cons(nx) = C_n^0 cos^n x - C_n^2 cos^{n-2} x(isinx)^2 + C_n^4 cos^{n-4} x(isinx)^4 + ... + C_n^m cos^{n-m} x(isinx)^m, (m \leq n). \tag{3}$$

Considering $sin^2 x = 1 - cos^2 x$, $cos(nx)$ may be translated into an expression only containing $cosx$, that is $cos(nx) = f_n^*(cosx)$, where

$$f_n^*(y) = C_n^0 y^n - C_n^2 y^{n-2}(1 - y^2) + C_n^4 y^{n-4}(1 - y^2)^2 + ... + i^m C_n^m y^{n-m}(1 - y^2)^{m/2}, (m \leq n). \tag{4}$$

Therefore, both slowly-ranging parameters can be written into

$$w_i = A_i cos(\Omega_i t) = A_i f_{n_i}^*(cos(\Omega t)) = A_i f_{n_i}^*(w(t)), (i = 1, 2). \tag{5}$$

In short, both w_1, w_2 can be indicated by the function of the single basic slow variable $w(t) = cos(\Omega t)$, i.e. $w_1 = A_1 f_1(w(t)), w_2 = A_2 f_2(w(t))$. And then the original system may be recognized as the coupling of a fast subsystem

$$\begin{cases} \dot{x_1} = \sigma(x_2 - x_1) + \varepsilon sgn(x_1), \\ \dot{x_2} = -x_2 - \delta x_3 + [r - x_4 + A_1 f_1(w(t))]x_1 + A_2 f_2(w(t)), \\ \dot{x_3} = \delta x_2 - x_3, \\ \dot{x_4} = -bx_4 + x_1 x_2, \end{cases} \tag{6}$$

and a slow subsystem $w(t) = cos(\Omega t)$.

Obviously, system (6) exists the only non-smooth interface $\Sigma := \{(x_1, x_2, x_3, x_4)|x_1 = 0\}$ due to the non-smooth term $\varepsilon sgn(x_1)$, which separates the vector field of the system into two individually smooth sub-regions, denoted by

$D_+ := \{(x_1, x_2, x_3, x_4)|x_1 > 0\}$ and $D_- := \{(x_1, x_2, x_3, x_4)|x_1 < 0\}$, each corresponding to a non-autonomous smooth subsystem $F_+(X)$ and $F_-(X)$.

Generalized equilibrium points of the system along with their stabilities are given below, and the regular bifurcations that may be involved in are analyzed. As a result of the discontinuity of the vector field, those irregular bifurcations on the non-smooth interface are explored with the help of differential inclusion theory[15].

3.1 Stability analysis of equilibrium points and conventional bifurcation analysis

It is easy to find the expression form of generalized equilibrium points by calculation, expressed by $E_\pm\left(x_0 \pm \frac{\varepsilon}{\sigma}, x_0, \delta x_0, \frac{\sigma x_0^2 \pm \varepsilon x_0}{\sigma b}\right)$, where x_0 always satisfies

$$\sigma^2 x_0^3 \pm 2\sigma\varepsilon x_0^2 + (\sigma^2\delta^2 b - \sigma^2 br - \sigma^2 bA_1 f_1(w(t)) + \sigma^2 b + \varepsilon^2)x_0 \mp \sigma br\varepsilon - \sigma b\varepsilon A_1 f_1(w(t)) - \sigma^2 bA_2 f_2(w(t)) = 0. \quad (7)$$

The stability conditions for generalized equilibria $E_\pm$ are determined by the eigenvalues of the corresponding Jacobi matrix. Notably, due to the symmetry of the system, the characteristic equations of two smooth subsystems $F_\pm(X)$ have the same form, i.e.

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \quad (8)$$

where $a_1 = \sigma + b + 2, a_2 = (1 - \frac{\sigma}{b})x_0^2 + (\frac{2\varepsilon}{\sigma} - \frac{\varepsilon}{b})x_0 + \frac{\varepsilon^2}{\sigma^2} + \delta^2 - \sigma A_1 f_1(w(t)) + \sigma b - \sigma r + 2\sigma + 2b + 1$,
$a_3 = (3\sigma + 1 + \frac{\sigma}{b})x_0^2 + (4\varepsilon + \frac{2\varepsilon}{\sigma} + \frac{\varepsilon}{b})x_0 + \frac{\varepsilon^2}{\sigma^2} + \frac{\varepsilon^2}{\sigma} - \sigma bA_1 f_1(w(t)) - \sigma A_2 f_2(w(t)) + \sigma\delta^2 + \delta^2 b - \sigma rb + 2\sigma b - \sigma r + \sigma + b$,
$a_4 = 3\sigma x_0^2 + 4\varepsilon x_0 + \frac{\varepsilon^2}{\sigma} - \sigma bA_1 f_1(w(t)) + \sigma\delta^2 b - \sigma rb + \sigma b$, and λ denotes the eigenvalue.

In accordance with Routh-Hurwitz criteria, when equilibria $E_\pm$ hold the conditions

$$a_1 > 0, a_1a_2 - a_3 > 0, a_1a_2a_3 - a_1^2a_4 - a_3^2 > 0, \quad (9)$$

the real parts of all roots of Eq. (8) are negative. Hence, the generalized equilibria $E_\pm$ are stable.

The changes of parameters will make equilibrium points no longer holding the stability conditions, leading various bifurcations due to the instability of equilibria. Next, the critical conditions under two possible conventional bifurcations are given in the following.

1.fold bifurcation

When parameters hold the following conditions

$$FB_0 : a_4 = 0(a_1 > 0, a_1a_2 - a_3 > 0, a_3 > 0), \quad (10)$$

it means fold bifurcation may emerge, causing jump phenomenon among equilibrium points.

2.Hopf bifurcation

If eigenvalues cross the imaginary axis, Hopf bifurcation might occur, and according to Hopf bifurcation theorem, parameters possess

$$HB_0 : a_1a_2a_3 - a_1^2a_4 - a_3^2 = 0(a_1 > 0, a_1a_2 - a_3 > 0, a_3 > 0). \quad (11)$$

In that way, the bifurcation could result in the instability of equilibrium and the emerge of limit cycle, whose oscillation frequency is

$$\omega_0^2 = \frac{a_3}{a_1}. \quad (12)$$

3.2 Non-smooth bifurcation analysis

As the discontinuity of the non-smooth interface $\Sigma := \{(x_1, x_2, x_3, x_4)|x_1 = 0\}$ at the vector field of system(6), different types of unconventional bifurcations may appear while the trajectory contacting with the interface. Complex dynamic behaviors of the trajectory at non-smooth interface Σ are focused on with the employment of differential inclusion theory, by introducing an auxiliary parameter $q \in [0, 1]$. At this point, the system (6) has the following form

$$\dot{X}(t) = \begin{cases} F_+(X), & q = 1, \\ qF_+(X) + (1-q)F_-(X), & 0 < q < 1, \\ F_-(X), & q = 0. \end{cases} \quad (13)$$

Table 1: Expressions of $f_1(w(t)$ and $f_2(w(t)$ under different excitation frequency values.

sets	Ω	$cos(\Omega t)$	Ω_1	Ω_2	$K=\frac{\Omega_1}{\Omega_2}$	$f_1(w(t)$	$f_2(w(t)$
1	0.005	w	0.005	0.01	2	w	$2w^2-1$
2	0.005	w	0.005	0.015	3	w	$4w^3-3w$
3	0.005	w	0.005	0.02	4	w	$8w^4-8w^2+1$
4	0.005	w	0.01	0.015	1.5	$2w^2-1$	$4w^3-3w$
5	0.005	w	0.01	0.025	2.5	$2w^2-1$	$16w^5-20w^3+5w$

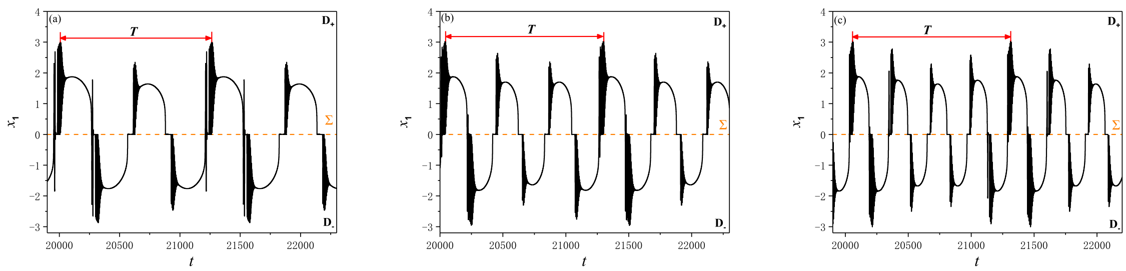

Figure 1: The time history of state variable x_1. (a)$K=2$; (b)$K=3$; (c)$K=4$.

It can be easily found that, the behaviors of trajectory are determined by the smooth subsystem $F_-(X)$ in smooth sub-region D_- when $q=0$; and if $q=1$, these behaviors are influenced by the other smooth subsystem $F_+(X)$ in smooth sub-region D_+; but when $0<q<1$, the trajectory will contact with the non-smooth interface Σ, causing sliding phenomena on the interface, whose dynamical behaviors are decided by $qF_+(X)+(1-q)F_-(X)$. Then the auxiliary parameter q could be described by calculation,

$$q=\frac{\varepsilon-\sigma x_{s2}}{2\varepsilon}, \tag{14}$$

where x_2 is the corresponding coordinate value when the trajectory touches with the interface.

4 Mechanism analysis of bursting oscillations

To further explore the dynamics of the system, we fixed parameter sets at $\sigma=2.8, \delta=1.5, r=5.3, b=1.1, \varepsilon=-1, A=2.5, A_1=0.5, A_2=2.5$ via numerical simulations and the frequency of the basic slow-variable function $w(t)=cos(\Omega t)$ is selected as $\Omega=0.005$. Hence, the Moivre formula is used to represent two slowly-changing parameters of system(6), thus achieving the separation of the fast and slow subsystems. Table 1. shows five sets of expressions about two frequencies with different values.

In the following portions, the influences on dynamic behaviors of system(6) under various excitation frequency ratios of integer or fractional are explored.

4.1 Bursting oscillations under integer excitation frequency ratio

Firstly, the time history of state variable x_1 for three different integer exciting frequency ratios $K=2, K=3, K=4$ are shown in Fig. 1 to explore the characteristics of bursting oscillation modes about system (6) with an integer ratio. It could be obviously found that the system exhibits periodical mixed-mode oscillations, but shows asymmetry, neither centrosymmetric about the origin nor axisymmetric. As frequency ratios K growing, the number of quiescent and spiking states significantly increases in a period, which leads to a richer dynamical behavior.

In order to analyze the bursting oscillations as well as the bifurcation mechanism of system (6), the transformed phase portraits along with the corresponding equilibrium curves are given in Fig. 2. As frequency ratio K increases, the number of fold bifurcation points, Hopf bifurcation points and the segment of equilibrium curves increase accordingly, which makes the bursting oscillations more complex and the dynamics more abundant. Notably, the number of fold bifurcation points and Hopf bifurcation points is always 2 times the corresponding frequency ratio, for instance, the number of fold

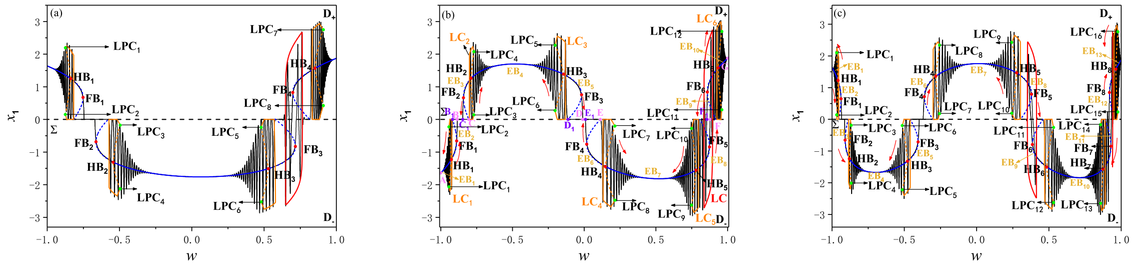

Figure 2: The overlap of the transformed phase portrait and the corresponding equilibrium curves. (a)$K = 2$; (b)$K = 3$; (c)$K = 4$.

and Hopf bifurcation points is 4 for $K = \frac{\Omega_2}{\Omega_1} = 2$, which is also a regular phenomenon in the case of integer frequency ratio.

Taking Fig. 2(b) as an example, the bursting oscillations of system(6) for $K = 3$ will be specifically discussed. Without loss of generality, assuming the trajectory starts at point A, it nearly moves along the stable equilibrium curve EB_1. Then, due to the slow passage effect by Hopf bifurcation point HB_1, the trajectory keeps moving forward until fold bifurcation point FB_1. It jumps to the non-smooth interface Σ at $B(w, x_1) = (-0.885, 0)$, corresponding $B_s(x_{s1}), x_{s2}, x_{s3}, x_{s4} = (0, -0.10, -0.891, 0.237)$ in traditional phase space. The auxiliary parameter q is computed $q|_{B_s} = (\varepsilon - \sigma x_{s2})/2\varepsilon|_{B_s} \approx 0.36 \in (0, 1)$ by Eq. (14), which means that the trajectory slides on the interface for a while, presenting special quiescent state. The trajectory is appealed by limit cycle LC_2 at point C, entering spiking state. As the limit cycle converges, the trajectory gradually moves to the equilibrium curve EB_4 and jumps to point D on the interface because of fold bifurcation point FB_3. Then, it oscillates along limit cycle LC_4 until the cycle converges to equilibrium curve EB_7. The effects of Hopf bifurcation point HB_5 and fold bifurcation point FB_5 make the trajectory skipping to point F. In the way, the trajectory is attracted by another stable limit cycle LC, into a large amplitude oscillation, crossing the interface and oscillating back and forth in two sub-regions. As the amplitude gradually decreases, the trajectory turns to be attracted by the limit cycle LC_6, and in the process of convergence of the cycle, the trajectory gradually converges to the stable equilibrium curve EB_{10}. The first half period of motion completes at the maximum slow variable $w = 1$, going through the process of $A \to B \to C \to D \to E \to F \to G$.

As time t slowly increases, the slow variable decreases, causing the trajectory to fold back toward the left and start the second half-cycle motion, and the trajectory returns to point A after six jumps $G \to F_1 \to E_1 \to D_1 \to C_1 \to B_1 \to A$ to complete a periodic motion. Although the trajectory is asymmetric, the mechanism is similar to that of the first half, so we will not repeat it here.

4.2 Bursting oscillations under fractional excitation frequency ratio

Next, the influences on oscillation modes when the frequency ratio is fractional are further explored. Fig. 3. shows the time history of state variable x_1 for $K = 1.5$ and $K = 2.5$, from which, we can find that the trajectory all exhibits centrosymmetric mixed-mode oscillations. This phenomenon is the biggest difference from the above situations.

The detailed processes of trajectory evolutions could be revealed by the overlap between the transformed phase portrait and equilibrium branches, as shown in Fig. 4. The explanations of the mechanism are similar to that described above and will not be expanded here. It could also be found that the number of bifurcation points as well as the number of the transitions between spiking and quiescent states of the trajectory increase with the ratio K.

When the frequency ratio is fractional or integer, the symmetry of the system is its most typical characteristic.

5 Conclusions

Based on a four-dimensional laser system, by introducing two periodic excitation terms, we mainly study various complex mixed-mode oscillations and the bifurcation mechanism of a non-smooth system under the combined action of excitations. Here, the Moivre formula is used to convert two excitation terms into the expression form of the same slow-variable parameter, so that only one slow variable exists in the original system. Thus the conventional fast-slow analysis is per-

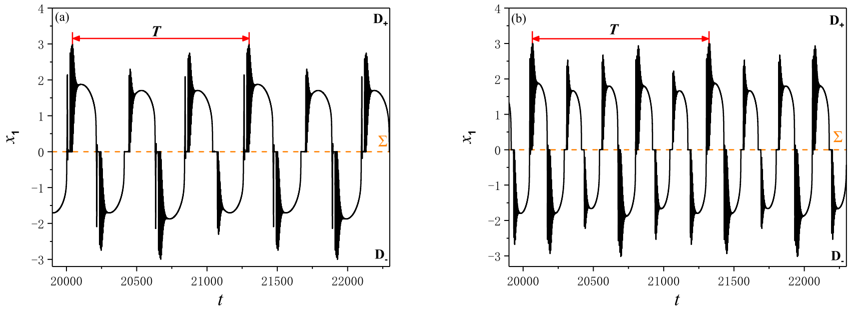

Figure 3: The time history of state variable x_1. (a)$K = 1.5$; (b)$K = 2.5$.

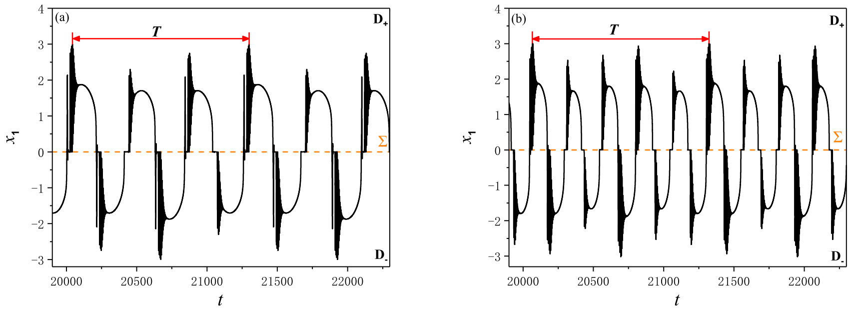

Figure 4: The overlap of the transformed phase portrait and equilibrium curves. (a)$K = 1.5$; (b)$K = 2.5$.

formed, including the stability analysis of equilibrium points of the fast subsystem and the critical conditions under two conventional bifurcations. Subsequently, the effects on bursting forms when the frequency ratio is integer or fractional are mainly explored. When the excitation frequency ratio is integer, the mixed-mode oscillations of the system are asymmetric, while if the ratio is fractional, the mixed-mode oscillations are centrosymmetric.

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